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ABSTRACT

This is a supplementary SMSG mathematics text for junior high school students. Key ideas emphasized are structure of arithmetic from an algebraic viewpoint, the real number system as a progressing development, and metric and non-metric relations in geometry. Chapter topics include sets, projective geometry, open and closed paths, finite differences, and formulas. (MP)

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MATHEMATICS FOR JUNIOR HIGH SCHOOL

SUPPLEMENTARY UNITS

SE 027 923

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SUPPLEMENTARY UNITS

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SUPPLEMENTARY UNIT 1

SETS

1-1. Introduction

You already are familiar with the word "set." A set of dishes is a collection of dishes. A set of dominoes is a collection, or group, of dominoes. In mathematics we use the word "set" to speak about any collection of any kind of thing. In your classroom there is a set of persons. There is also a set of noses, and a set of desks. You may notice that there is a relation between the set of persons in the room and the set of noses in the room.

The language of sets is very useful in describing all sorts of situations. How is the set of pupils in your class related to the set of boys in the class? Compare the number in each following three sets:

the set of pupils in your class,
the set of boys in your class, and
the set of girls in your class.

The following three sets are related in a different way:
the set of redheads,
the set of baboons, and
the set of redheaded baboons.

In this chapter we are going to study relations between sets, and ways in which we can combine sets to obtain new ones. We shall find it convenient to invent some new words and symbols. It might be a good idea to review those chapters before reading further.

1-2. Sets, Their Members and Their Subsets

Sets and Their Members

When we speak of a set as a collection of things, we do not mean that the things are all together in one place or time. The set of all living women is a widely distributed set. You will meet members of this set all over the world. The set of all presidents of the United States has as members George Washintton and Dwight D. Eisenhower, among others. Name other members of this set.

The "things" may not be objects which you can touch or see. The set of all Beethoven symphonies does not contain any concrete objects. You may have heard some of its members. The set of all school orchestras in the United States is a set whose members are themselves sets of pupils. The set of classes in your school is another set whose members are sets. It is different from the set of all students in classes in your school. Which of these sets has more members: Are there more classes or students in your school?

Sometimes we define a set by listing its members. Your teacher might appoint a committee to be in charge of the mathematical exhibits in your class. She may say, "The members of the Exhibits Committee shall be Lenore, Muriel, Dick and Al."

We often name a set which is defined in this way by listing names of its members and enclosing them in braces:

Exhibits Committee = {Lenore, Muriel, Dick, Al}.

We use the symbol "ϵ" (Greek letter epsilon) to mean "is a member of." Thus we can express the fact that Lenore is on the committee by writing

Lenore ϵ Exhibits Committee.

We could state the definition of the committee like this:

> $x \epsilon$ Exhibits Committee if and only if x represents Lenore or x represents Muriel or x represents Dick or x represents Al.

Another way to describe a set is to state the membership requirements. These are conditions that something must satisfy in order to get into the set. The set of persons in your classroom has a very simple membership requirement. The object x is in the set if x is a person in your classroom, and only then. The set of common multiples of 4 and 6 is the set of all x for which it is true that x is a multiple of 4 and x is a multiple of 6. You might imagine each object in the universe applying for membership in this set. If the object is not even a whole number, then we throw it out immediately. If it is a whole number, we divide it by 4. If the remainder is zero, we then divide the number by 6 and see whether 6 is a factor. If x passes this test, too, then x gets its membership card in the set. If it fails any of the tests, we reject it.

We sometimes call the members of a set "elements of the set." You are an element of the set of mathematics students.

Property

You begin to see that for a particular set to be clearly defined there must be some scheme or device for determining whether or not a given element is in the set. Usually a set is described in terms of some property, or properties, which its elements have in common. For example, the set C may be thought of as the pupils in your class. The common property is that each element is a member of your class. Again, you may consider set B as the set of boys in your class. The element of this set contain two properties in common: (1) the elements are all in your class, and (2) the elements are all boys. Sometimes a set is described simply by enumerating the elements. For example, the set of even whole numbers may be described by writing: 0, 2, 4, 6, 8, 10 - - - . What is the common property in this set?

Exercises 1-2-a

1. List a common property or properties of the following sets:
   - (a) {Sue, Jane, Dorothy, Mildred}.
   - (b) {Washington, Jackson, Eisenhower}.
   - (c) {1, 3, 5, 7, 9, 11}.
   - (d) {12, 24, 36, 48}.

2. Translate the following mathematical sentences into English.
   (a) Tom $\in$ {Carl, Jim, Tom, Robert}.
   (b) $6 \in \{0, 2, 4, 6, 8, 10, ---\}$.
   (c) If $X \in$ {Tom, Carl, Bob, Jim} then X represents Tom, or X represents Carl, or X represents Bob, or X represents Jim.

3. Which of the following are true?
   (a) $4 \in \{3, 7, 10, 4\}$.
   (b) lion $\in$ {baboon, tiger, dog, lion}.
   (c) $X \in \{8, 14, 17, 28\}$ where X is a multiple of 6.
   (d) $X \in \{1, 2, 3, 4, 5, 6, ---\}$ where X is a counting number.
   (e) Washington, D.C. $\in$ {Alabama, Alaska, Arizona, ···, West Virginia, Wisconsin, Wyoming}.

4. List the members of the following sets:
   (a) The set of X such that X is a factor of 12 and 30.
   (b) The set of X such that X plays a violin, or X plays the viola, or X plays the cello.
   (c) The set of X such that X is a whole number.
   (d) The set of X such that X is one of the U.S. Presicents since 1930.

Subsets

Consider the set of major league baseball teams in New York in 1959. This set has one member, the New York Yankees Baseball Club. Its one member is itself a set, among whose members are Mickey Mantle and Yogi Berra. The set whose only member is a certain object is not the same as that object. The symbol $\{3\}$ is a name for the set whose only member is 3.

The set, or team, of New York Yankees is a subset of baseball players. Every member of the team is a baseball player. In symbols, we write: If $X \in$ Yankees, then $X \in$ the set of baseball players.

You have been introduced to a new word: that of subset. Let us consider another example. Suppose in a class of 25 pupils there are 3 pupils whose first name begins with "S." You can then say that these 3 pupils form a subset of the class. Again, consider the set of even counting numbers: 2, 4, 6, 8, 10, - - - . This set can be considered as a subset of the counting numbers: 1, 2, 3, 4, 5, 6, - - - .

Suppose the set of pupils in your class whose first names begin with "S" is {Sam, Susan, Sally}. The subsets of this set may be listed as follows: {Sam}, {Susan}, {Sally}, {Sam, Susan}, {Sam, Sally}, and {Susan, Sally}. Sometimes we say that a set is a subset of itself.

Definition:

A set R is a subset of a set S if every element of R is an element of S.

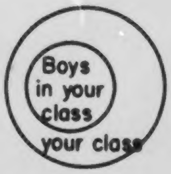
It is necessary, at times, to talk about the relationship of a subset to a set, or the relationship of a set to another set. We say, for example, that the set of even counting numbers (which is a subset of the counting numbers) is contained in the set of counting numbers. To write this in mathematical language we use the symbol "$\subset$", which is read "is contained in." You can now write: $\{2, 4, 6, 8, - - -\} \subset \{1, 2, 3, 4, 5, 6, - - -\}$. Sometimes the symbol "$\supset$" is also used. This is read "contains." You can now also write:

$$\{1, 2, 3, 4, 5, - - -\} \supset \{2, 4, 6, 8, - - -\},$$

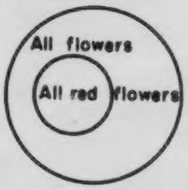
which reads: The set of counting numbers contains the set of even counting numbers. Let the set of your class be called "C" and the set of boys in your class be called "B". You can then write:

$$B \subset C, \text{ or}$$
$$C \supset B.$$

You may be helped in this study by use of diagrams. A mathematician always draws figures or diagrams when possible. The diagrams used below are called "Venn" diagrams. Consider again the example $B \subset C$. We sketch the following



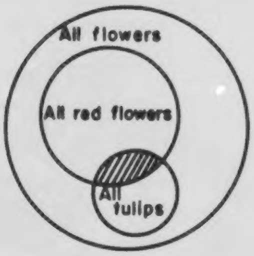
This illustrates that the set of boys in your class is contained in the set of your class. Again:



illustrates that the set of all red flowers is contained in the set of all flowers. Let the set of all red flowers be called R and the set of all flowers be called F. The relationship of R and F can then be written as:

$$R \subset F, \text{ or}$$
$$F \supset R.$$

Consider the following Venn diagram:



This diagram indicates that the set of all red flowers belongs to the set of all flowers. It also indicates that the set of all tulips belongs to the set of all flowers. Let the set of all tulips be called T. The above relationships may now be expressed as:

$$R \subset F, \text{ and}$$
$$T \subset F.$$

What can you say about the relationship of set R and set T? You would certainly have to say that some tulips are red and are thus contained in the set R, but you certainly cannot say that $T \subset R$ is true. Give some thought to this situation for a while.

Exercises 1-2-b

1. Translate the following mathematical sentences into English:

(a) If $X \in \{\text{Red flowers}\}$, then $X \in$ the set of all flowers.

(b) $M \subset N$, and $N \supset M$.

(c) $\{1, 3, 5, 7, 9, 11 - - -\} \subset \{1, 2, 3, 4, 5, 6, - - -\}$.

2. Write all possible subsets of the set: $\{4, 5, 6\}$.

3. Translate the following English sentences into mathematical sentences.

(a) The set $\{12, 20, 32\}$ is contained in the set of all whole numbers.

(b) The set of the Great Lakes contains the set of Lake Huron and Lake Michigan.

(c) The set of {Hoover, Truman} is contained in the set of all U.S. presidents since 1920.

4. Draw a Venn diagram to illustrate the following:

(a) The set of the Hudson and Ohio Rivers is contained in the set of all rivers in the United States.

(b) The set of tigers, lions, and baboons is contained in the set of all animals.

(c) The set of 16, 36, and 40 is contained in the set of all counting numbers which are multiples of 4.

(d) The set of 6, 1/2, 3/8 is contained in the set of all rational numbers.

5. Which of the following are true and which are false?

(a) $\{\text{Al, Tom}\} \supset \{\text{Al, Bob, Jack, Tom}\}$.

(b) $\{\text{Sam, Sue}\} \subset \{\text{Slim, Tom, Bob, Sally}\}$.

(c) The set of all yellow roses is contained in the set of all yellow flowers.

(d) $\{28, 56, 112\} \subset$ the set whose elements are multiples of 4 and also of 7.

6. Given three sets A, B, and C. If $A \supset B$ and $B \supset C$, does $A \supset C$? Illustrate your answer with a Venn diagram.

1-3. Operations with Sets

Union

Suppose the set: {Bill, Jim, Tom, Sam} are the boys of a class who play in the band. Call this set B. Let the set: {Sam, Tom, Carl} be the boys in the same class who have red hair. Call this set R. Now if we combine these two sets we would get the set: {Bill, Carl, Jim, Tom, Sam}. This would be the set consisting of all elements which belong to set B, or to set R, or to both sets. We call this the union of two sets. The symbol used is: "$\cup$". We can now write:

$$\{\text{Bill, Jim, Tom, Sam}\} \cup \{\text{Sam, Tom, Carl}\} = \{\text{Bill, Carl, Jim, Tom, Sam}\}.$$

If we call the union of these two sets C, then you can write:

$B \cup R = C$, and it is read: B union R equals C.

The combining of two sets in this manner is called an operation. Before working some problems let us consider another matter which was introduced by writing $B \cup R = C$.

Equality of Sets

We say that two sets are equal if and only if each element of one is also an element of the other. Suppose we have two sets A and B: If $A \subset B$ and $B \subset A$ then we can say $A = B$.

For example, suppose that in your class there are only four redheaded pupils which we shall call set R, and furthermore, these four redheaded pupils are the only ones having their birthdays in January, which we shall call set J. We can write:

$$R \subset J \text{ and } J \subset R, \text{ hence } R = J.$$

Consider again: $B \cup R = C$. If we can write $(B \cup R) \subset C$ and $C \subset (B \cup R)$, then we can say: $B \cup R = C$. After some thought you should see that this is a true statement. Instead of saying that two sets are equal, we sometimes say they are identical. This is a good expression since we can say that two sets are equal if and only if every element of each is an element of the other.

Properties

1. Consider again the two sets, B and R. Do you suppose that

$$B \cup R = R \cup B?$$

Let us investigate:

$$\begin{aligned} B \cup R &= \{\text{Bill, Jim, Tom, Sam}\} \cup \{\text{Sam, Tom, Carl}\} \\ &= \{\text{Bill, Carl, Jim, Tom, Sam}\}. \\ R \cup B &= \{\text{Sam, Tom, Carl}\} \cup \{\text{Bill, Jim, Tom, Sam}\} \\ &= \{\text{Bill, Carl, Jim, Tom, Sam}\}. \end{aligned}$$

You see, then, that $B \cup R = R \cup B$. Does this recall to you what you learned about the "commutative" property? With a little thought on the union concept, you should see that for any two sets M and N, $M \cup N = N \cup M$, and the commutative property is true for sets under the operation of union.

2. Doyou think the following is true?

$$A \cup (B \cup C) = (A \cup B) \cup C.$$

Let $A = \{1, 2, 3\}$; $B = \{1, 4\}$; $C = \{2, 5, 6\}$.

Then: $A \cup (B \cup C) = \{1, 2, 3\} \{1, 2, 4, 5, 6\}$

$= \{1, 2, 3, 4, 5, 6\}$,

and: $(A \cup B) \cup C = \{1, 2, 3, 4\} \cup \{2, 5, 6\}$

$= \{1, 2, 3, 4, 5, 6\}$.

You see, then, that in our example: $A \cup (B \cup C) = (A \cup B) \cup C$. This should recall to mind the associative property. With some thought you should see that under the operation of union the associative property is true for sets.

Exercises 1-3-a

1. (a) If set $M = \{\text{Red, Blue, Green}\}$ and set $N = \{\text{Blue, Yellow, White}\}$, find $M \cup N$.

   (b) Is $M \cup N = N \cup M$? Why?

2. Let A be the set of even counting numbers; B the set of odd counting numbers; and C the set of all counting numbers.

   (a) Is $A \cup B = C$? Why?

   (b) Is $A \subset C$? Why?

   (c) Is $A \subset B$? Why?

   (d) Is $A \cup B = B \cup A$? Why?

   (e) Does $B \supset A$? Why?

   (f) Draw a Venn diagram to illustrate $B \subset C$.

   (g) Is $A = B$? Why?

3. Given three sets R, S, and T.
   (a) Is $(R \cup S) \cup T = R \cup (S \cup T) = T \cup (R \cup S)$? Why?
   (b) Suppose $(R \cup S) \subset T$ and $T \subset (R \cup S)$, then is $R \cup S = T$? Why?

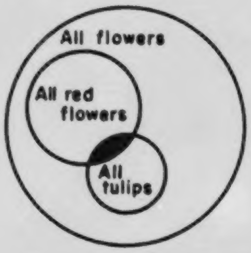
4. Let C be the set of pupils in your class, S be the set of pupils in your school, and X be the only redheaded pupil in your class. Discuss the following as to whether or not they are true.
   (a) $X \in S$ (b) $C \subset S$ (c) $C = S$
   (d) $S \subset C$ (e) $X \in C$ (f) $S \supset C$
   (g) Is X a subset of C? Of S?
   (h) Is C a subset of S?

5. (a Consider two concentric circles. Let X be the set of points within a circle whose radius is 4 units and Y be the set of points within a circle whose radius is 2 units. Draw a Venn diagram to show: $X \cup Y$.
   (b) Is $X \subset Y$, or $Y \subset X$? After giving your answer complete the statement: _____ is a subset of _____.

Intersection

Another operation with sets is that of intersection. Do you recall this operation from Chapter 4? You no doubt remember that the symbol for intersection is "$\cap$". Consider sets A and B. If we now write: $A \cap B$, it is read "A intersection B."

The intersection of two sets is the set of all elements which belong to both sets. For example, let set A be {Tom, Sue, Carl, Joan}, and set B be {Sam, Sue, Tom, Sally}. Then $A \cap B = \{Sue, Tom\}$. Do you remember the following Venn diagram we had several pages back?



You remember a question was raised about the relationship of R and T, where R was the set of all red flowers and T was the set of tulips. You can now see that the shaded part of the diagram is $R \cap T$. This situation presents us with another set which we have not mentioned. Are there any yellow tulips in set R?

Null Set

At times we have a set which is said to be empty. Such a set is sometimes called the "null set." For example, the set of yellow tulips contained in the set of all red flowers is an example of a null set. Suppose there are no redheaded pupils in your class then the set of redheaded pupils in your class is a null set. Another example is the set all voters who have their legal residence in Washington, D.C. We shall use the symbol "$\emptyset$" (the Greek letter phi, pronounced "fee") to designate the null set. We say that $\emptyset$ is a subset of every set.

Properties

1. Given two sets M and N: Is it true that $M \cap N = N \cap M$? Let M be $\{1, 2, 3, 4\}$ and N be $\{3, 4, 5, 6\}$, then $M \cap N = \{3, 4\}$ and $N \cap M = \{3, 4\}$. In view of your previous study you are led to see that the commutative property applies under the operation of intersection of sets.

2. In a similar manner, given three sets R, S, and T, it can be shown that the associative property holds. We would then have: $R \cap (S \cap T) = (R \cap S) \cap T$. Select an example of your own and see if you get a true result.

3. Are you reminded of anything by the following, where R, S and T are three sets?

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T).$$

Let $R = \{1, 2, 7\}$, $S = \{1, 3, 4\}$ and $T = \{2, 3, 5\}$.

Then
$$\begin{aligned} R \cup (S \cap T) &= \{1, 2, 7\} \cup (\{1, 3, 4\} \cap \{2, 3, 5\}) \\ &= \{1, 2, 7\} \cup \{3\} \\ &= \{1, 2, 3, 7\} \end{aligned}$$

and $(R \cup S) \cap (R \cup T) =$
$$\begin{aligned} &= (\{1, 2, 7\} \cup \{1, 3, 4\}) \cap (\{1, 2, 7\} \cup \{2, 3, 5\}) \\ &= \{1, 2, 3, 4, 7\} \cap \{1, 2, 3, 5, 7\} \\ &= \{1, 2, 3, 7\}. \end{aligned}$$

This illustrates the distributive property of union with respect to intersection of sets. In working with sets we have two forms of this property. We have just studied one form: namely, $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$. The other form is: $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$, which is the distributive property, of intersection with respect to union of sets. This is somewhat different from what you studied in working with the counting numbers in Chapter 3. There was only one form of the distributive property: namely, multiplication with respect to addition.

Exercises 1-3-b

1. Given the three sets: $A = \{\text{boy, girl, chair}\}$, $B = \{\text{girl, chair, dog}\}$ and $C = \{\text{chair, dog, cat}\}$.
   (a) Find $A \cap B$.
   (b) Show that $A \cap C = C \cap A$.
   (c) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
   (d) Show that $A \cap (B \cap C) = C \cap (A \cap B)$.

2. Where $\emptyset$ represents the null set, and H is any other set, is the following true? $\emptyset \cup H = H \cup \emptyset$. Explain your answer. Is $\emptyset \quad H = H$? Explain your answer. Under the operation of union of sets, what name may be applied to $\emptyset$?

3. Let R represent the set of points on the line segment $\overline{AB}$, and S represent the set of points on another line segment $\overline{CD}$.
   (a) If $R \cap S = \emptyset$, then what is true about the two line segments?
   (b) If $R \cap S \neq \emptyset$, then what is true about the two line segments?

4. Are there any similarities between the symbols "$\cup$" and "$\cap$", and the symbols " + " and " $\cdot$ "? Explain your answer.

5. Draw a Venn diagram to illustrate the intersection set of all members of the band in your school and all the pupils in your class.

6. Show by use of a figure the intersection set of two intersecting circular regions.

7. (a) Let E be the set of even counting numbers: $\{2, 4, 6, 8, \cdots\}$. What must be the set F so that
$$E \cup F = C,$$
when C is the set of all counting numbers?
   (b) What is the set of $E \cap F$?

8. Given two sets A and B:
   (a) If $A \subset B$, is it true that $A \cup B = B$? Explain your answer.
   (b) If $A \subset B$, is it true that $A \cap B = A$? Explain.

1-4. Order, One-to-One Correspondence, the Number of a Set, and Counting

Order

In many situations the order in which we write the elements of a set is immaterial. For example, set A: {Bill, Tom, Sam}, can be written as {Tom, Sam, Bill}, or as {Sam, Bill, Tom}, just as well as in the original. Under our definition of equality, all three of these sets are equal. At times, however, the order is important. For example, the name William Thomas is not the same as Thomas William. If we wrote these two names as a set: {William, Thomas}, then under our present framework, we could just as well write the set as: {Thomas, William}, and the two sets would be equal, or identical. An ordered set is one wherein there is an element which is the first term, another element which is a second term, and so on. When we wish to indicate that the elements of a set are ordered, we shall use the symbol: "( )". If we now write the set composed of the elements Thomas, William in the form: (Thomas, William) it is not equal to the set: (William, Thomas), because the set is ordered with the element Thomas in the first position and the element William in the second position. A set of two elements written in this manner is sometimes called an ordered pair. You had some contact with ordered pairs when you made graphs in Chapter 11. A set such as: (a, b, c) may be referred to as an ordered triple. This idea may be extended to many more then 3 elements. For example, the ordered set of the first n counting numbers: (1, 2, 3, 4, 5, 6, ···, n), would give us an "n-tuple" where n may be any counting number. This idea will be used in the section on Counting.

Ordered pairs are very useful in many branches of mathematics. When you study a course called Analytical Geometry, you will deal with ordered pairs such as (1, 4), (6, 2), (12, 15), for example.

Consider the set of people in line before the box office of a theater. Is order important in this situation? If you should try to move ahead of someone already in line, you would be made to understand, rather quickly, the importance of order in this case. There are people who consider order important enough to take a bed roll and sleep near a box office, so as to be well up in line when the office opens. Some baseball fans do this for the World Series. Can you think of other similar situations?

As you know, the following is a true statement:

$$\{1, 2, 3\} = \{1, 3, 2\}.$$

On the other hand, $(1, 2, 3) \neq (1, 3, 2)$, because these are ordered sets.

One-to-One Correspondence

One basic study of sets deals with the comparison of two or more sets to see whether or not they are equally numerous. This is done by matching the elements of the sets. In the opening pages of Chapter 2 you read that in the long ago a shepherd probably kept account of his sheep by having a notched stick - a notch for each sheep and a sheep for each notch. With this arrangement he could tell whether or not any sheep were missing by comparing, or matching, the set of notches with the set of sheep. If all sheep were present, we could say there was a one-to-one correspondence between the set of sheep and the set of notches.

Consider your class. Suppose there is the same number of seats in your classroom as there are pupils in your class. When all the pupils are present, then the set of seats and the set of pupils are in one-to-one correspondence. In other words, the two sets are equally numerous. If all pupils are present and seated in their assigned seats, then your teacher can tell at a glance that there is perfect attendance for the day. Without much more than a glance she can tell how many are absent, if some are not present. How does she do this? What can you say with respect to one-to-one correspondence of the following:

1. {1, 2, 3, 4}; {O, X, ∧, V}; {A, B, C, D}.
2. {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}; {a, b, c, d, e, f, g, h, i, j}.
3. {the number of fingers on one hand};
   {the number of symbols in a base five system};
   {the number of players on a boys' basketball team}.

We are now in a position to state a general principle with respect to sets and one-to-one correspondence as follows:

> Given two sets A and B. These two sets are said to be in one-to-one correspondence if we can pair, or match, the elements of A and B such that each element of A pairs with one and only one element of B, and in the same matching process each element of B pairs with one and only one element of A. This principle may be stated more precisely in the following way:
>
> Let A and B be sets. There is a one-to-one correspondence between A and B if there exists a collection H of ordered pairs with the following properties:
>
> 1. The first term of each pair of H is an element of A,
> 2. The second term of each pair of H is an element of B,

3. Each element of A is a first term of exactly one pair of H,
4. Each element of B is a second term of exactly one pair of H.

In Problem 2 above let A be the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, and B the set {a, b, c, d, e, f, g, h, i, j}. The set H would look like this:

(1, a)
(2, b)
(3, c)
(4, d)
(5, e)
(6, f)
(7, g)
(8, h)
(9, i)
(10, j).

Unless the concept of order is to be taken into consideration, the matching process may be done in more than one way. Consider set A: {Bill, Tom, Sam}, and set B: {Ann, Jane, Susan}. Since these sets have only three elements, we can see at a glance that there is a one-to-one correspondence between them. The matching process, however, can be done in six ways. Two of them are as follows:

Set A		Set B
Bill	⟷	Ann
Tom	⟷	Jane
Sam	⟷	Susan

Set B		Set B
Bill	⟷	Ann
Tom	⟷	Susan
Sam	⟷	Jane

Figure 1-4

The symbol " ↔ " simply means, for example, that Bill is matched with Ann, and Ann is matched with Bill.

Let us consider the elements of these two sets again, and write the sets as follows:

A: (Bill, Tom, Sam), B: (Ann, Jane, Susan).

The notation indicates the two sets are now ordered. Of course, we can still match the elements in six ways. If, however, we want to preserve the order, the elements can be matched in only one way as follows:

Bill ↔ Ann
Tom ↔ Jane
Sam ↔ Susan.

Equivalence

You remember when we talked about the equality of sets, we said that two sets were equal, or identical, if and only if every element of each is an element of the other. For example,

$$\{1, 2, 3\} = \{1, 3, 2\}$$

because the two sets contain the same elements. The concept of one-to-one correspondence introduces a new concept of equality, that of *equivalence*. We say that two sets which are in one-to-one correspondence are equivalent. We shall indicate this fact by using the symbol " ⟷", which was used in matching the elements of sets. For example: {Bill, Tom, Sam} ⟷ {Ann, Jane, Susan}. Again, given two sets A and B, if we write: A ⟷ B, we mean that there is a one-to-one correspondence

Exercises 1-4-a

1. Construct tables similar to those of Figure 1-4 to show the additional four ways in which the two sets may be matched.

2. By observing Figure 1-4 and the additional tables you made in Problem 1, you will notice that Bill $\leftrightarrow$ Ann twice. Without making tables can you determine the number of possible matchings for the sets: $\{1, 2, 3, 4\}$ and $\{a, b, c, d\}$?

3. Determine whether the following are true or false. Use examples to illustrate your answers.
   (a) Identical sets are also equivalent.
   (b) Equivalent sets are also identical.
   (c) Equivalent sets may be identical.
   (d) Equivalent sets are never identical.
   (e) Identical sets are never equivalent.

4. Construct a matching table for the following sets so that order will be preserved: (1, 2, 3, 4, 5, 6), (x, y, t, a, b, c).

5. Suppose you buy a carton of a dozen eggs. Is it necessary to count the eggs in order to tell whether or not you have a dozen? Why?

6. Given two sets x and y. If $x \subset y$ and $y \subset x$, can we say that the two sets are in one-to-one correspondence? Explain.

7. Are there more points on an arc of a circle than on its subtended chord? Explain your answer.

The Number of a Set

Given the sets: {1, 2, 3, 4} and {0, 1, Λ, V}. You notice that there is a one-to-one correspondence between them. In addition you see that the sets are composed of 4 elements. In fact, any two sets which are in one-to-one correspondence have the same number of elements. Sets, however, will vary in the number of elements which they contain. This may vary all the way from zero, the null set, to an infinity of elements. The word "infinity" is not new to you, because you will remember that there are an infinite number of points on a line, or again, an infinite number of whole numbers. A set containing an infinite number of elements is called an *infinite set;* otherwise, the set is called a *finite set*. Since sets vary in the number of elements they contain, we can, then, assign a number to a set. We can only assign the same number, however, to those sets which have a one-to-one correspondence between them. In this discussion we shall consider only finite sets.

When we wish to talk about the number of a set we shall use the following notation: $n(A)$. This is read: "the number of set A." More briefly it is at times read: "n of A." For the sets:

$$\{1, 2, 3, 4\} \text{ and } \{0, 1, \wedge, V\},$$

we can now write:

$$n(\{1, 2, 3, 4\}) = n(\{0, 1, \wedge, V\}).$$

Counting

The use of the counting numbers: (1, 2, 3, 4, 5, 6, 7, ···), gives us a basic sequence which we may consider as the numbers of finite sets. Every counting number, then, may be considered as the number of the set of all counting numbers up to and including it.

Counting can be considered as a method of matching between any finite set and a subset of the counting numbers. Let us designate the set of counting numbers as C. Further, let us label the subsets of C as C_1, C_2, C_3, ··· ; where $C_1 = \{1\}$, $C_2 = \{1, 2\}$, $C_3 = \{1, 2, 3\}$, and so on. As an example, let us count the set A composed of {Sam, Carl, Tom, Jack}.

Set A: {Sam, Carl, Tom, Jack}

Set C: {1, 2, 3, 4, 5, 6, 7, ··· }.

By matching you see that set A matches with subset C_4 of the set C. Since $n(C_4) = 4$, then $n(A) = 4$.

Consider set A: {1, 2, 3, 4}, and set B: {5, 6, 7}, which are said to be disjoint. Two sets are said to be disjoint if they contain no elements in common. Now do you remember the expression $A \cup B$? Applying the operation we get a new set: {1, 2, 3, 4, 5, 6, 7}. Upon matching this new set with C, you note that it is C_7. So $n(A \cup B) = n(C_7) = 7$. Let us consider the problem through another example: Given the disjoint sets, M: {a, b, c, d}, and N: {e, f, g}. Now $M \cup N = \{a, b, c, d, e, f, g\}$. Upon matching this new set with

C, you notice that it is also C_7. Hence we have:
$n(M \cup N) = n(C_7) = 7$.

Do you now notice that the number of the union of the two disjoint sets may be considered as the sum of the number of the sets?

Exercises 1-4-b

1. What is the number name of the following sets?
   (a) $\{1, 2, 3, 4, 5, 6\}$.
   (b) $\{a, b, c, d\}$.
   (c) {bird, dog, cat, chair, horn}.
   (d) $\{1, \times, *, \square, V, \wedge\}$.
   (e) Which of the above sets have the same number?

2. Suppose a set R matches subset T of another set S. What can you say about the number of R in relationship to the number of S?

3. Considering only finite sets, if set M matches set N, and set N matches set R, what is the relationship of set M to set R?

4. How does the number of the set of automobiles being driven at this moment compare with the number of the set of their steering wheels?

5. By matching the sets C_{12} and C_7, show that $7 < 12$.

6. Given two sets A: {Bob, Sue, Tom, Joe} and B: {cat, dog, chair}. Find the set $A \cup B$. Now match the union of these sets with C and determine the number of the union set.

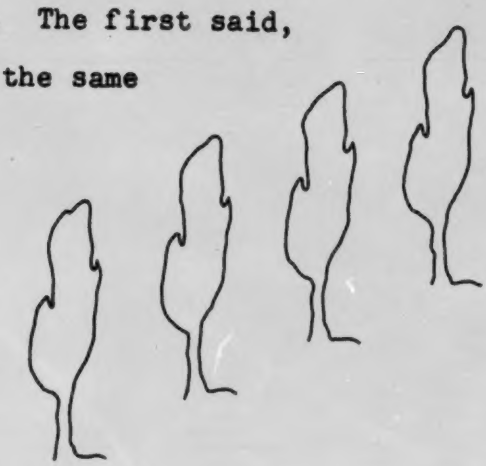
7. Given the two disjoint sets M: {1, 2, 3, 4}, and N: {5, 6, 7, 8 9}. Find M N and determine $n(M \cup N)$ by comparing it with C.

SUPPLEMENTARY UNIT 2

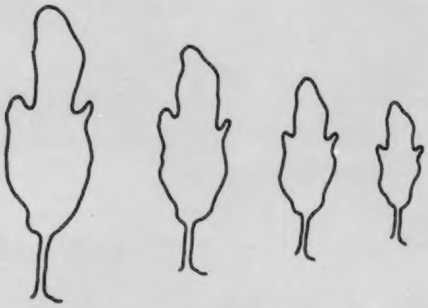
SPECIAL FIGURES IN PROJECTIVE GEOMETRY

2-1. Geometry and Art

In a certain park there is a row of poplar trees. They are evenly spaced, and all the same size and shape. Two boys wanted to draw a picture of them. The first said, "I know that these trees are all the same size. I know that there is the same distance between any two adjacent ones. This is how I will draw them."



The other said, "The trees further off look smaller to me, and even though I know they are not smaller I will draw them as I see them." Which of their pictures do you like better?



The second boy used the idea of perspective. This is a very important idea in art if we are interested in drawing things the way they really look to us. It is the idea used in giving depth to a picture.

Of course, not all artists have wanted todo this. In ancient Egyptian art, for example, it was the rule to draw the pharaoh larger than anyone else in a picture, and the sizes of other people were made to depend on their importance.

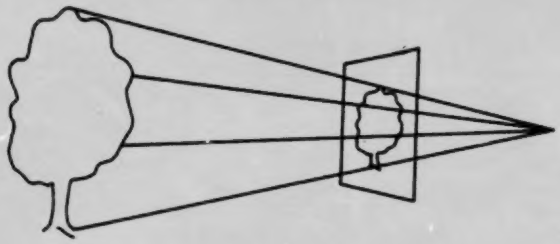
Not until the end of the Middle Ages did artists make serious systematic efforts to understand perspective. At that time they became greatly interested in learning the rules that would help them picture realistically the world about them. This period, which historians call the Renaissance, was a time of great development in science and learning as well as art. It was a time of new ideas and of a new interest in understanding the laws of nature. It was a time of experiment.

One of the artists of this period was Leonardo da Vinci. Though we remember him best for his paintings, he had a wide range of interests. Among other things he tried to design a way man could fly. He believed that a knowledge of science and mathematics is an essential tool for the artist.

An artist who did a great deal of work in developing rules of perspective was Albrecht Dürer. In some of his drawings we can see the way in which he studied these problems. You can find examples of them in Mathematics in Western Culture, by Morris Kline. This book contains many other pictures you will also find interesting.

A mathematician, Girard Desargues, wrote a book about the ideas of geometry that would be useful in connection with the study of perspective. He was the originator of what is called projective geometry.

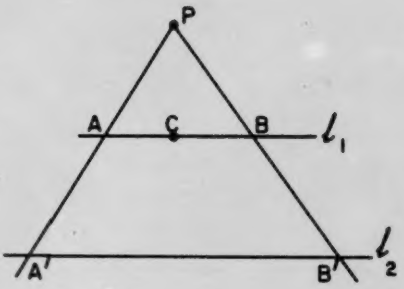
The word "projective" can be understood if we think about drawing a picture. In drawing a tree, you can think of a line extending from each point you see to your eye.



Each line intersects the plane of your canvas in a point. The points in the picture thus match the points of the tree that we see. A geometer says that the picture (the set of points) on the canvas is a projection of the set of points of the tree.

Here is another example that will help you understand the sort of problems that occur in projective geometry. Suppose there is a triangular rose bed in a garden. Suppose an artist draws this rose bed several times. Perhaps he draws it first as seen from a point in the garden. Next he draws it as seen from the top of a high tower. Perhaps he tries other locations as well. He will find that in his pictures the rose bed is always triangular. He will find, however, that the triangle has different shapes depending on where he stands. He has discovered: The projection of a triangle is a triangle. Later we will see another discovery that can be made about this situation.

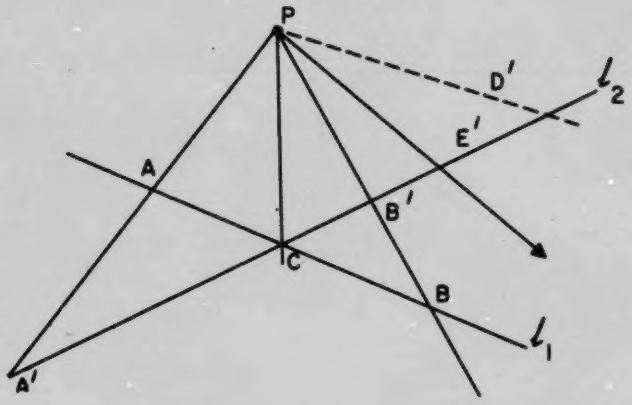
Projective Geometry in a Plane -- One-to-One Correspondences of Point Sets



In this figure, lines ℓ_1 and ℓ_2 are parallel. Lines drawn from point P intersect lines ℓ_1 and ℓ_2. One such line intersects ℓ_1 in A and ℓ_2 in A'. Another intersects ℓ_1 in B and ℓ_2 in B'. The figure gives us a way of matching the points on ℓ_1 with the points on ℓ_2. To find the point on ℓ_2 that matches C, for example, we would draw the line through C and P. The point where it intersects ℓ_2 is the point that matches C.

This matching of one set (the points on ℓ_1) with another set (the points on ℓ_2) is called a one-to-one correspondence, as we know. We have found a one-to-one correspondence between the points on ℓ_1 and the points on ℓ_2. (Of course, if we used some other point in place of P we would find another one-to-one correspondence between the points on ℓ_1 and those on ℓ_2. The two point sets can be matched in many different ways.)

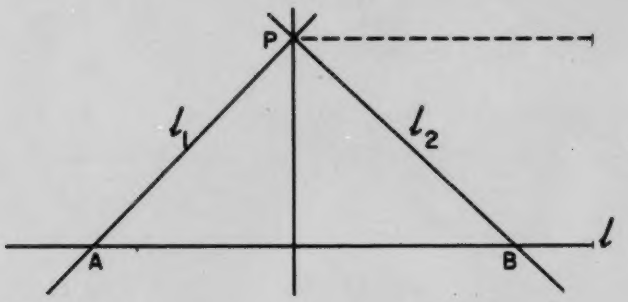
Did you wonder why we chose parallel lines for ℓ_1 and ℓ_2? Let us see what would happen if we did not. In the figure ℓ_1 and ℓ_2 are not parallel. We can still draw lines through P cutting ℓ_1 and ℓ_2. Point A on ℓ_1 corresponds to point A' on ℓ_2. Point B corresponds to B'. Point C is a special point. It belongs to both the set of points on ℓ_1 and the set of points on ℓ_2. A line through P that intersects ℓ_1 in C also intersects ℓ_2 in C. In the correspondence between points on ℓ_1 and points on ℓ_2 the point C matches itself.



It looks as though we have once again a one-to-one correspondence between the points on ℓ_1 and the points on ℓ_2. But we need to stop and think very carefully. We need to remember that there is one line through P that is parallel to ℓ_1. Suppose this line (the dotted line in the figure) intersects ℓ_2 in the point D'. D' is a point on ℓ_2, but our system does not give any point on ℓ_1 that matches it. Points on ℓ_2 that are very close to D' match points that are very far out on ℓ_1. E' is one such point.

There is also a line through P that is parallel to ℓ_2. So there is also a point on ℓ_1 that has no matching point on ℓ_2. We have discovered: Our system gives us a way of matching all the points except one on ℓ_1 with all the points except one on ℓ_2.

Here is another example of a one-to-one correspondence between sets. This figure shows some of the elements of the set of lines through P.



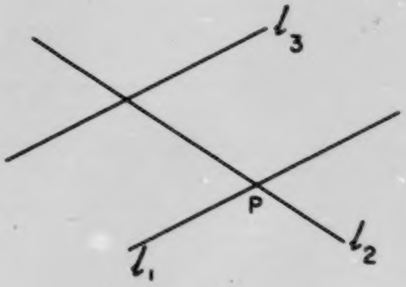
Each of the lines through P in the figure intersects the line ℓ in a point. The figure shows a way of matching elements of the set of lines through P with elements of the set of points on ℓ. The line ℓ_1 matches the point A. The line ℓ_2 corresponds to point B.

Again, however, we need to be careful. There is one line through P that is parallel to ℓ. This line does not have a matching point on ℓ. We see that: To each point on ℓ corresponds a line through P. To each line through P except one there corresponds a point on ℓ.

The Idea of Ideal Points

These examples will help you understand an idea that is very useful in projective geometry. It is the idea of an ideal point on a line.

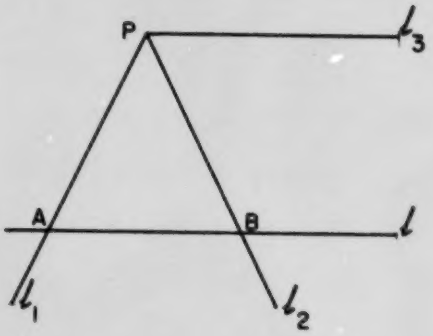
In projective geometry we do not use the term "parallel lines." Instead, we use the term "lines that intersect in an ideal point." We think of each line as containing one and only one ideal point, as well as the usual points we are accustomed to thinking about. In order to be quite clear, we can call the usual points *real points*. When we adopt this new language, we can say that *any* two lines in a plane meet in a point of some sort. In the figure, ℓ_1 and ℓ_2 meet in the real point P. ℓ_1 and ℓ_3 meet in an ideal point. Formerly we would have said they are parallel. The two statements mean the same thing.



In our new language, the set of all points on a line is made up of all the real points and, in addition, the ideal point

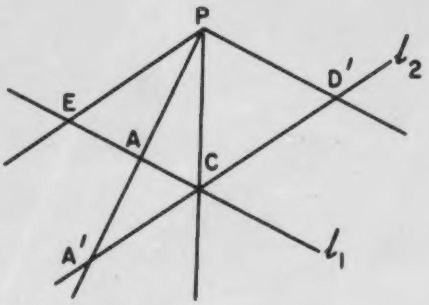
Let us use this new vocabulary to describe the one-to-one correspondences which we have already studied. As we do so, we will find that it is a very convenient language for describing these situations.

In this figure we can now say that there is a one-to-one correspondence between the set of *all* lines through P and the set of *all* points on ℓ. Line ℓ_1 corresponds to the real point A. Line ℓ_2 corresponds to the real point B. Line ℓ_3 we now say, intersects

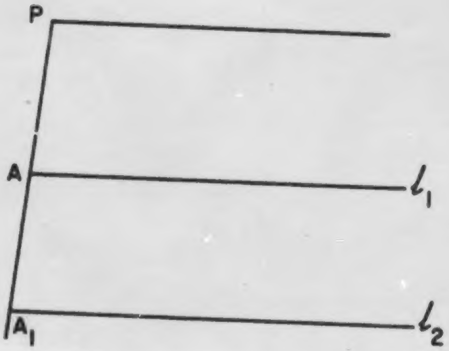


line ℓ in the ideal point of ℓ. It corresponds to the ideal point on ℓ.

In this figure we can now say that there is a one-to-one correspondence between the set of all points on ℓ_1 and the set of all points on ℓ_2. The real point A on ℓ_1 corresponds to the real point A' on ℓ_2. The real point C belongs to the set of all points on ℓ_1 and to the set of all points on ℓ_2. It corresponds to itself. The point D' on ℓ_2 corresponds to the ideal point on ℓ_1. The point E on ℓ_1, corresponds to the ideal point on ℓ_2. (Remember that we now say that each line contains an ideal point. The line through P and E intersects ℓ_2 in the ideal point.)



In this figure ℓ_1 and ℓ_2 intersect in an ideal point. There is a one-to-one correspondence between the set of all points on ℓ_1 and the set of all points on ℓ_2. The line through P and A intersects ℓ_1 and ℓ_2 in corresponding real points. The line through P parallel to ℓ_1 and ℓ_2 intersects ℓ_1 and ℓ_2 in an ideal point. This ideal point is an element of the set of all points on ℓ_1. It is also an element of



the set of all points on ℓ_2. It corresponds to itself in the one-to-one correspondence.

We have introduced the idea of ideal point so that every pair of lines intersects in a point, that is, "two lines determine a point." What about the statement, "Two points determine a line," by which we mean that there is exactly one line through any two points? This is certainly true in the geometry that we are used to, that is, for two real points. But is it still true for projective geometry? Suppose A is an ideal point and B is a real point. From our definition of ideal points, A must be on some ℓ_1, since from our familiar geometry there is exactly one line through A and B. Thus through any pair of points we can draw exactly one line except when both of the points are ideal points. And we can remedy this deficiency by defining an "ideal line" on which all the ideal points lie. This fits in very nicely because then the ideal line will intersect every other line in just one point -- its ideal point.

One big advantage of projective geometry is that not only do two points determine a line but two lines determine a point. This symmetrical arrangement is very convenient.

The language of ideal points is new to you. Like any new language, it seems difficult until one is accustomed to it. The examples illustrate its advantages. When we use the idea of ideal points we do not have to consider parallel lines as exceptions to our descriptions.

You will understand better how the idea of ideal points originated if you think about railroad tracks. When we draw railroad tracks we draw them as though they come together far away. The idea of ideal point is suggested by the way parallel lines sometimes appear to meet when we draw objects in perspective.

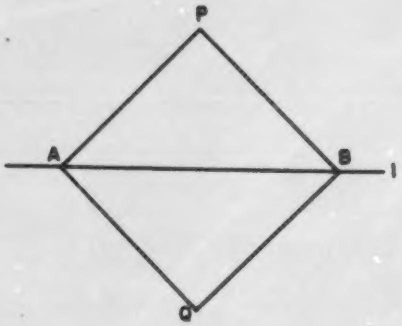
Of course, if you are building a railroad track the idea of ideal points is not useful at all. When we build railroad tracks we need to know, for example, that all the ties that lie between the tracks are the same length. The idea of length is studied in _metric_ geometry. Metric geometry uses the idea of measurement. Projective geometry does not; this is why we say that projective geometry is _non-metric_.

You may feel that ideal points seem unnatural. But you should remember that all points, lines, and planes are ideas. They are ideas that are developed because they are interesting and useful for some purpose.

Exercises 2-1

1. Draw two parallel lines. Call them ℓ_1 and ℓ_2. Mark a point P between them. By drawing lines through P, find a one-to-one correspondence between the points on ℓ_1 and the points on ℓ_2. Label the points in your drawing, and name three pairs of corresponding points.

2. Mark points P and Q. Draw a line ℓ between them, as



in the figure. The figure shows a way of matching the set of lines through P with the set of lines through Q. To the line through A and P corresponds the line through A and Q. The line through B and P is matched with the line through B and Q. In this way we can find a ____________ __________________ between the set of lines through P and the set of lines through Q. Draw three other pairs of lines illustrating this statement.

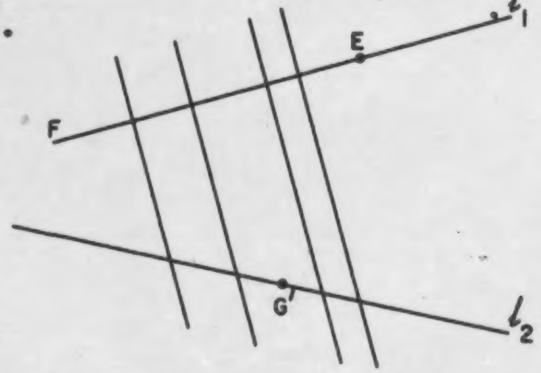
3. In Exercise 2, is there a line which belongs both to the set of lines through P and the set of lines through Q?

4. In Exercise 2, which line through P corresponds to the line through Q parallel to ℓ? This line through P intersects ℓ in an ______________________ .

5. Explain the meaning of the following statement: If P is any real point and ℓ is any line not passing through P, there is exactly one line which passes through P and through the ideal point on ℓ.

6. In this figure four of the lines are parallel.

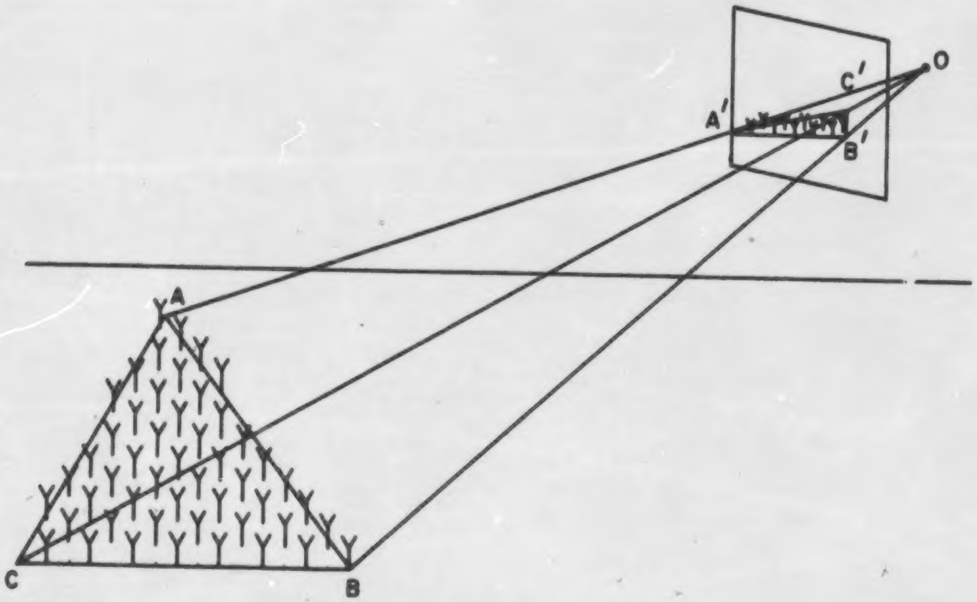
   (a) Four of the lines intersect in an ________________
   ________________.



(b) The figure shows a system for finding a one-to-one correspondence between the points of ℓ_1 and the points of ℓ_2. Find the points corresponding to E, F, and G'.

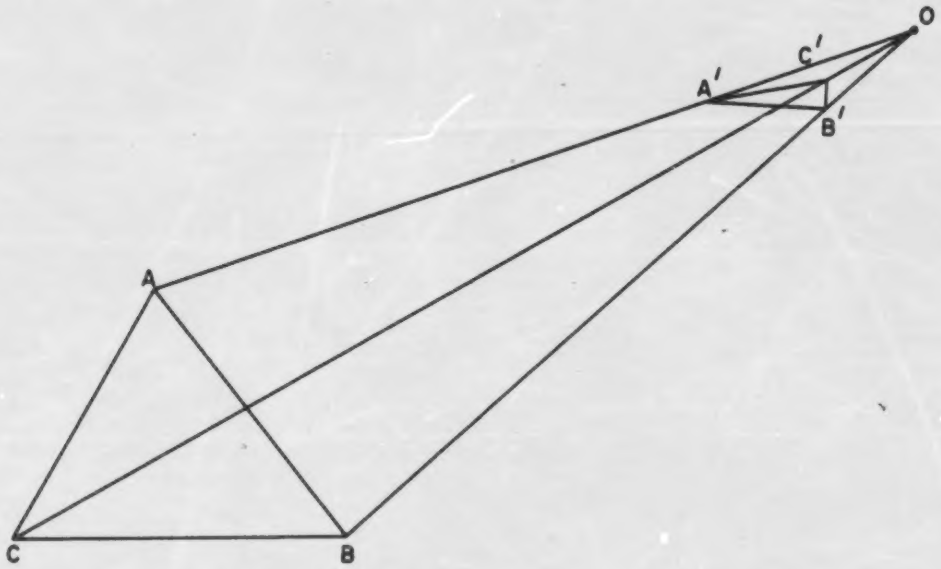
Desargues' Theorem

One of the most interesting ideas in projective geometry is that contained in Desargues' Theorem. In order to understand it, let us think again about a situation we considered earlier. Let us think about an artist who is drawing a triangular rose bed Suppose that he is drawing his picture as he sees it from a tower high above a garden. Here is a sketch that shows the two triangles -- the boundary of the rose bed and the picture of it on his canvas. Each point on the rose bed is matched with a point on the canvas triangle.



In the sketch the vertices of the rose bed are called A, B, and C. In the artist's picture, the matching vertices are labeled A', B', C'. The three lines joining matching vertices all meet in point O -- the eye of the artist. The two triangles are said to be in perspective.

We can draw two triangles in the same plane that are in perspective. In the following figure two such triangles have been drawn. Again, the vertices of one triangle are matched with the vertices of the other. Again, the lines joining corresponding vertices meet in a point.

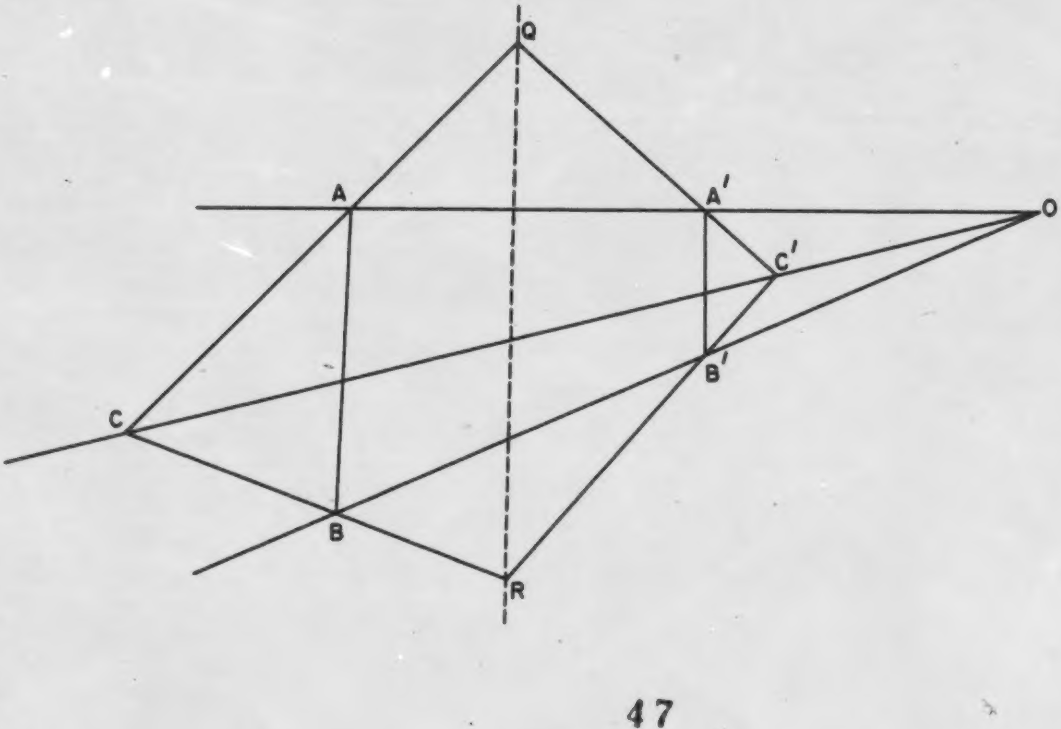


Exercise 2-2

Copy this figure carefully. Extend AB and A'B' until they intersect. Do the same thing with AC and A'C'. Do the same thing with BC and B'C'. You have found three intersection points. Label them P, Q, and R. Do you notice anything about these three points? They should all lie on the same line.

A boy said, "I wonder whether this will always be true if I extend the sides of a pair of triangles in perspective." He tried it several times. It appeared to be true each time. Of course it was sometimes difficult to be sure, because he needed to extend the lines a long way to find the intersection points. He decided, however, that it was probably always true that the three points of intersection were on the same line.

"But what about this figure?" asked another boy. "In my triangles, AB and A'B' have the same direction. When I extend them I get parallel lines. There is no point of intersection."



"I notice something about the figure you have drawn, though," the first boy replied. "Those two lines are parallel to the line through Q and R. I think that this is another place where the idea of ideal point might be useful. We could say that the three points of intersection are all on the same line, but now one of the points is an ideal point."

He was right. If it is true that

(a) two triangles are in perspective, and

(b) each pair of corresponding sides, extended, has a point of intersection,

then the three points of intersection all lie on a line.

If, however, there is at least one pair of sides with the same direction, so that these sides, when extended, form parallel lines, then we have an exceptional case. The exceptional cases can be conveniently described by the idea of ideal point.

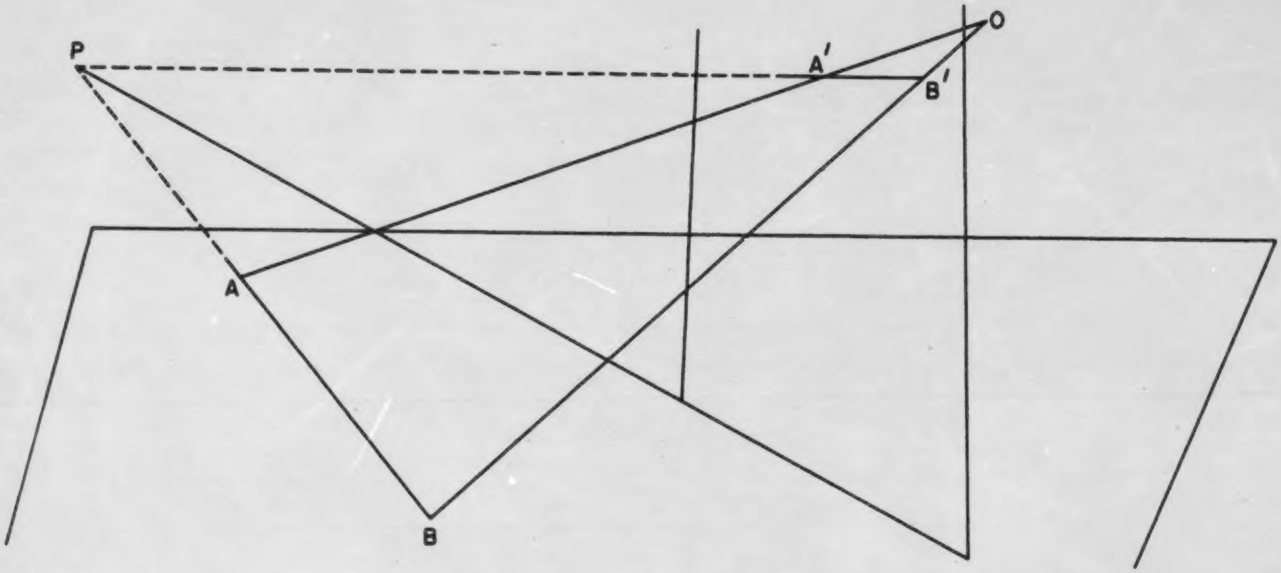
Of course, the second boy was not satisfied with leaving the matter at this. He wondered why the three intersection points all were on the same line. Perhaps you wonder too. If you do, you will be interested in knowing the way we prove that the points are always on a line. A proof makes us sure the statement is true -- a good proof also makes us understand better the reason.

Let us again think about the garden and the picture. Let us suppose that:

(a) the plane of the garden and the plane of the picture are not parallel (this is the way we drew the figure).

(b) none of the pairs of corresponding sides have the same direction.

Look at the line through A and A' and the line through B and B'. This figure will help you see the lines.



These two lines intersect at O. When we have a pair of intersecting lines, we can think about the plane they both lie in. The line through A and B is in this plane; so is the line through A' and B'. We supposed that these lines did not have the same direction. We know that two lines in the same plane that do not have the same direction meet, so we can be sure that these lines meet. P, of course, is the point where they meet.

Now let us think about where P is. P is on the line through A and B. This line is on the plane of the garden. So P must be on the plane of the garden. P is also on the line through A' and B', which is on the plane of the canvas. So P is also on the plane of the canvas. Now we can put these two facts together and say: P is on the intersection of two planes -- the plane of the canvas and that of the garden. The intersection of these two planes is a line.

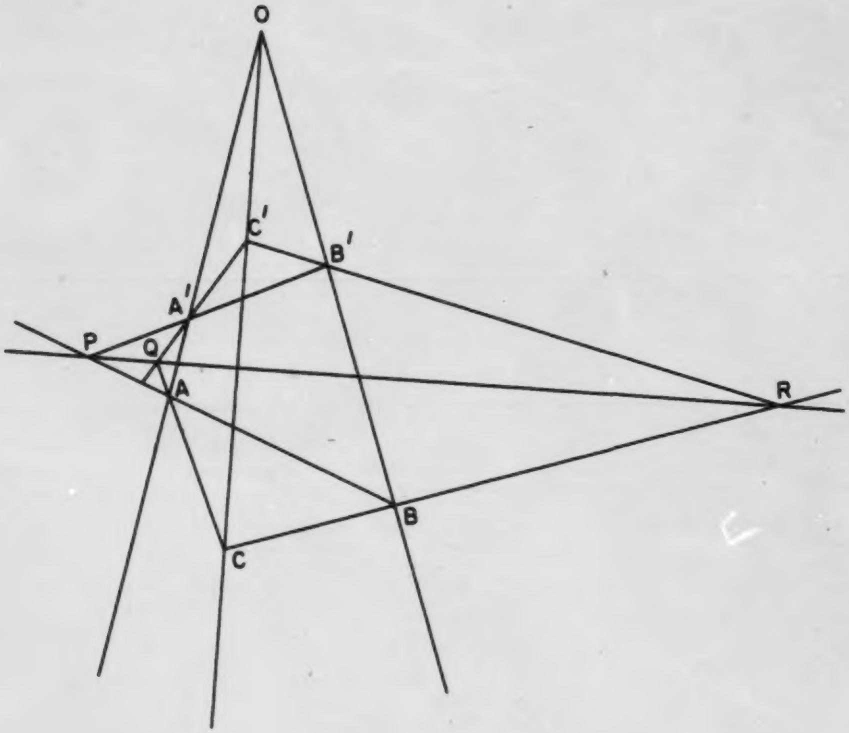
Now we have proved that P is on the line of intersection of a certain pair of planes. We can prove in precisely the same way that the line through B and C and the line through B' and C' meet in a point, which we have labeled Q. We can also prove, by exactly the same reasoning as that used in the case of P, that R is on the line of intersection of the plane of the canvas and the plane of the garden. Then we can reason the same way about the point R.

So we can see that P, Q, and R all lie on the same line -- the line where our two planes intersect.

Now we have proved our fact for two triangles that are in different (and not parallel) planes.

It is more difficult to prove that it is true when the two triangles are in the same plane. We can see, however, that if we took a picture of the garden and the canvas, we would really have two triangles in perspective in the same plane, and that the points of intersection of the pairs of corresponding sides of the triangles would all be on a line. If you were more familiar with the use of geometric reasoning in rather complicated figures, you would not find it difficult to use this idea in constructing a complete proof.

Points and Lines in Desargues' Theorem



In the figure we see that there are 10 main points: the vertices of the two triangles, the point O, and the three intersection points P, Q, and R. There are also 10 main lines: the sides of the triangle extended, the lines through corresponding vertices of the triangles, and the line on which lie P, Q, and R. By checking the figure you can see that --

(a) through each of the main points there are three of the special lines, and

(b) on each of the special lines there are three special points.

The figure for Desargues' theorem could be used for a very "democratic" committee diagram, where, by "democratic" we mean that in certain respects each committee member is treated like every other one. We could let each of the ten points correspond to a person and each of the ten lines correspond to a committee. If a certain point is on a certain line, then the corresponding person would be on the corresponding committee. Then

1. Each committee has three members and each person is on three committees.
2. Each pair of committees has exactly one person in common and each pair of persons is on exactly one committee.

Exercises 2-3

1. Draw several figures illustrating Desargues' theorem.

2. One of the remarkable aspects of the figure for Desargues' theorem is that each point and each line play exactly the same role. For example, we might think of A as the "beginning" point in place of O and one triangle could be taken to be COB. Since the third point on AC is Q, the third point on AO is A', and the third point on AB is P, the second triangle must be QA'P. Then the points of

intersection of corresponding sides of the two triangles should be on a line. Find the line.

3. Follow through the steps in Exercise 2 starting with the point P.

4. The following converse of Desargues' theorem also holds: If ABC and A'B'C' are two triangles and if the points P, Q, R are defined as the intersections of the pairs AB, A'B'; AC, A'C'; BC, B'C' lie on a line, then AA', BB', CC' are concurrent. Draw a figure which shows this.

5. (Brainbuster) Designate seven points by the numbers: 1, 2, 3, 4, 5, 6, 7. Call the set of three points 1, 2, 4 a "line ℓ_1" and so on according to the following table:

Line	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
Points	1,2,4	2,3,5	3,4,6	4,5,7	5,6,1	6,7,2	7,1,3

Show that each point lies on three lines. Is it true that each pair of points determines a line? Is it true that each pair of lines determines a point? Draw a figure which shows this. (You cannot make all the lines straight and one will have to jump over another.)

SUPPLEMENTARY UNIT 3

REPEATING DECIMALS AND TESTS FOR DIVISIBILITY

3-1. Introduction

This monograph is for the student who has studied a little about repeating decimals, numeration systems in different bases, and tests for divisibility (casting out the nines, for instance) and would like to carry his investigation a little further, under guidance. The purpose of this monograph is to give this guidance; it is not just to be read. You will get the most benefit from this material if you will first read only up to the first set of exercises and then without reading any further do the exercises. They are not just applications of what you have read, but to guide you in discovery of further important and interesting facts. Some of the exercises may suggest other questions to you. When this happens, see what you can do toward answering them on your own. Then, after you have done all that you can do with that set of exercises, go on to the next section. There you will find the answers to some of your questions, perhaps, and a little more information to guide you toward the next set of exercises.

The most interesting and useful phase of mathematics is the discovery of new things in the subject. Not only is this the most interesting part of it, but this is a way to train yourself to discover more and more important things as time goes on. When you learned to walk, you needed a helping hand, but you really had not learned until you could stand alone. Walking was not new to mankind -- lots of people had walked before -- but it was new to you. And whether or not you would eventually discover places in your walking which no man had ever seen before, was unimportant. It was a great thrill when you first found that you could walk, even though it looked like a stagger to other people. So, try learning to walk in mathematics. And be independent -- do not accept any more help than is necessary.

3-2. Casting out the Nines

You may know a very simple and interesting way to tell whether a number is divisible by 9. It is based on the fact that a number is divisible by 9 if the sum of its digits is divisible by 9 and the sum of its digits is divisible by 9, if the number is divisible by 9. For instance, consider the number 156782. The sum of its digits is $1 + 5 + 6 + 7 + 8 + 2$ which is 29. But 29 is not divisible by 9 and hence the number 156782 is not divisible by 9. If the second digit had been a 3 instead of 5, or if the last digit had been 0 instead of 2, the number would have been divisible by 9 since the sum of the digits would have been 27 which is divisible by 9. The test is a good one because it is easier to add the digits than to divide by 9. Actually we could have been lazy and instead of dividing 29 by 9, use the fact again, add 2 and 9 to get 11, add the 1 and 1 to get 2 and see that since 2 is not divisible by 9, then the original six digit number is not divisible by 9.

Why is this true? Merely dividing the given number by 9 would have tested the result but from that we would have no idea why it would hold for any other number. We can show what is happening by writing out the number 156,782 according to what it means in the decimal notation:

$$1 \times 10^5 + 5 \times 10^4 + 6 \times 10^3 + 7 \times 10^2 + 8 \times 10 + 2 =$$
$$1 \times (99999 + 1) + 5 \times (9999 + 1) + 6 \times (999 + 1) +$$
$$7 \times (99 + 1) + 8 \times (9 + 1) + 2.$$

Now by the distributive property, $5 \times (9999 + 1) = 5 \times 9999 + 5 \times 1$ and similarly for the other expressions. Also we may rearrange the numbers in the sum since addition is commutative. So our number 156,782 may be written

$$1 \times (99999) + 5 \times (9999) + 6 \times (999) +$$
$$7 \times (99) + 8 \times 9 + (1 + 5 + 6 + 7 + 8 + 2).$$

Now 99999, 9999, 999, 99, 9 are all divisible by 9, the products involving these numbers are divisible by 9 and the sum of these products is divisible by 9. Hence the original number will be divisible by 9 if $(1 + 5 + 6 + 7 + 8 + 2)$ is divisible by 9. This sum is the sum of the digits of the given number. Writing it out this way shows that no matter what the given number is, the same principle holds.

Exercises 3-2

1. Choose four numbers and by the above method test whether or not they are divisible by 9. When they are not divisible by 9, compare the remainder when the sum of the digits is divided by 9 with the remainder when the number is divided by 9. Could you guess some general fact from this? If you can, test it with a few other examples.

2. Given two numbers. First, add them, divide by 9 and take the remainder. Second, find the sum of their remainders after each is divided by 9, divide the sum by 9 and take the remainder. The final remainders is the two cases are the same. For instance, let the numbers be 69 and 79. First, their sum is 148 and the remainder when 148 is divided by 9 is 4. Second, the remainder when 69 is divided by 9 is 6 and when 79 is divided by 9 is 7; the sum of 6 and 7 is 13, and if 13 is divided by 9, the remainder is 4. The result is 4 in both cases. Why are the two results the same no matter what numbers are used instead of 69 and 79? Would a similar result hold for a sum of three numbers? (Hint: write 69 as 7 9 + 6.)

3. If in the previous exercise we divided by 7 instead of 9, would the remainders by the two methods for division by 7 be the same? Why or why not?

4. Suppose in Exercise 2 we considered the product of two numbers instead of their sum. Would the corresponding result hold? That is, would the remainder when the product of 69 and 79 is divided by 9 be the same as when the product of their remainders is divided by 9? Would this be true in general? Could they be divided by 23 instead of 9 to give a similar result? Could similar statements be made about products of more than two numbers?

5. Use the result of the previous exercise to show that 10^{20} has a remainder of 1 when divided by 9. What would its remainder be when it is divided by 3? By 99?

6. What is the remainder when 7^{20} is divided by 6?

7. You know that when a number is written in the decimal notation, it is divisible by 2 if its last digit is divisible by 2, and divisible by 5 if its last digit is 0 or 5. Can you devise a similar test for divisibility by 4, 8, or 25?

8. In the following statement, fill in both blanks with the same number so that the statement is true:
A number written in the system to the base twelve is divisible by _____ if its last digit is divisible by _____. If there is more than one answer, give the others, too. If the base were seven instead of twelve, how could the blanks be filled in? (Hint: one answer for base twelve is 6.)

9. One could have something like "decimal" equivalents of numbers in numeration systems to bases other than ten. For instance, in the numeration system to the base seven, the septimal equivalent of $5(1/7) + 6(1/7)^2$ would be written $.56_7$ just as the decimal equivalent of $5(1/10 + 6(1/10)^2$ would be written $.56_7$ in the decimal system. The number $.142857142857 \cdots$ is equal to 1/7 in the decimal system and in the system to the base seven would be written $.1$. On the other hand, $.1 = (.04620462 \cdots)_7$. What numbers would have terminating septimals in the numeration system to the base 7? What would the septimal equivalent of 1/5 be in the system to the base 7? (Hint: remember that if the only prime factors of a number are 2 and 5, the decimal equivalent of its reciprocal terminates.)

10. Use the result of Exercise 3 to find the remainder when $9 + 16 + 23 + 30 + 37$ is divided by 7. Check your result by computing the sum and dividing by 7.

11. Use the results of the previous exercises to show that $10^{20} - 1$ is divisible by 9, $7^{108} - 1$ is divisible by 6.

12. Using the results of some of the previous exercises if you wish, shorten the method of showing that a number is divisible by 9 if the sum of its digits is divisible by 9.

13. Show why the remainder when the sum of the digits of a number is divided by 9 is the same as the remainder when the number is divided by 9.

3-3. Why Does Casting out the Nines Work?

First let us review some of the important results shown in the exercises which you did above. In Exercises 2, you showed that to get the remainder of the sum of two numbers, after division by 9, you can divide the sum of their remainders by 9 and find its remainder. Perhaps you did it this way (there is more than one way to do it; yours may have been better). You know in the first place that any natural number may be divided by 9 to get a quotient and remainder. For instance, if the number is 725, the quotient is 80 and the remainder is 5. Furthermore, $725 = 80 \times 9 + 5$ and you could see from the way this is written that 5 is the remainder. Thus, using the numbers in the exercise, you would write $69 = 7 \times 9 + 6$ and $79 = 8 \times 9 + 7$. Then $69 + 79 = 7 \times 9 + 6 + 8 \times 9 + 7$. Since the sum of two numbers is commutative, you may reorder the terms and have $69 + 79 = 7 \times 9 + 8 \times 9 + 6 + 7$. Then, by the distributive property, $69 + 79 = (7 + 8) \times 9 + 6 + 7$. Now the remainder when $6 + 7$ is divided by 9 is 4 and $6 + 7$ can be written $1 \times 9 + 4$. Thus $69 + 79 = (7 + 8 + 1) \times 9 + 4$. So, from the form it is written in, we see that 4 is the remainder when the sum is divided by 9. It is also the remainder when the sum of the remainders, $6 + 7$, is divided by 9.

Writing it out in this fashion is more work than making the computations the short way but it does show what is going on and why similar results would hold if 69 and 79 were replaced by any other numbers, and, in fact, we could replace 9 by any other number as well. One way to do this is to use letters in place of the numbers. This has two advantages. In the first place it helps us be sure that we did not make use of the special properties of the numbers we had without meaning to do so. Secondly, we can, after doing it for letters, see that we may replace the letters by any numbers. So, in place of 69 we write the letter $\underline{a}$, and in place of 79, the letter $\underline{b}$. When we divide the number $\underline{a}$ by 9 we would have a quotient and a remainder. We can call the quotient the letter q and the remainder, the letter r. Then we have

$$a = (q \times 9) + r$$

where r is zero or some natural number less than 9. We could do the same for the number b, but we should not let q be the quotient since it might be different from the quotient when $\underline{a}$ is divided by 9. We here could call the quotient q' and the remainder r'. Then we would have

$$b = (q' \times 9) + r'.$$

Then the sum of a and b will be

$$a + b = (q \times 9) + r + (q' \times 9) + r'.$$

We can use the commutative property to have

$$a + b = (q \times 9) + (q' \times 9) + r + r'$$

and the distributive property to have

$$a + b = (q + q') \times 9 + r + r'.$$

Then if $r + r'$ were divided by 9, we would have a quotient which we might call q'' and a remainder r''. Then $r + r' = (q'' \times 9) + r''$ and

$$a + b = (q + q') \times 9 + (q'' \times 9) + r''$$
$$= (q + q' + q'') \times 9 + r''.$$

Now r'' is zero or less than 9 and hence it is not only the remainder when $r + r'$ is divided by 9 but also the remainder when $a + b$ is divided by 9. So as far as the remainder goes, it does not matter whether you add the numbers or add the remainders and divide by 9.

The solution of Exercise 4 goes the same way as that for Exercise 2 except that we multiply the numbers. Then we would have

$$\begin{aligned} 69 \times 79 &= (7 \times 9 + 6) \times (8 \times 9 + 7) \\ &= 7 \times 9 \times (8 \times 9 + 7) + 6 \times (8 \times 9 + 7) \\ &= 7 \times 9 \times 8 \times 9 + 7 \times 9 \times 7 + 6 \times 8 \times 9 + 6 \times 7. \end{aligned}$$

The first three products are divisible by 9 and by what we showed in Exercise 2, the remainder when 69×79 is divided by 9 is the same as the remainder when $0 + 0 + 0 + 6 \times 7$ is divided by 9. So in finding the remainder when a product is divided by 9 it makes no difference whether we use the product or the product of the remainders.

If we were to write this out in letters as we did the sum, it would look like this:

$$a \times b = (q \times 9 + r) \times (q' \times 9 + r')$$

$$= q \times 9 \times q' \times 9 + q \times 9 \times r' + r \times q' \times 9 + r \times r'.$$

Again each of the first three products is divisible by 9 and hence the remainder when $a \times b$ is divided by 9 is the same as when $r \times r'$ is divided by 9.

We used the number 9 all the way above, but the same conclusions would follow just as easily for any number in place of 9, such as 7, 23, etc. We could have used a letter for 9 also but this seems like carrying it too far.

There is a shorter way of writing some of the things we had above. When letters are used, we usually omit the multiplication sign and write ab instead of $a \times b$ and $9q$ in place of $9 \times q$. Hence the last equation above could be abbreviated to

$$ab = qq'9 \times 9 + qr'9 + rq'9 + rr'$$

or

$$ab = 9 \times 9qq' + 9qr' + 9rq' + rr'.$$

But this is not especially important right now.

So let us summarize our results so far: The remainder when the sum of two numbers is divided by 9 (or any other number) is the same as the remainder when the sum of the remainders is divided by 9 (or the same other number). The same procedure holds for the product in place of the sum.

These facts may be used to give quite a short proof of the important result stated in Exercise 13. Consider again the number 156,782. This is written in the usual form:

$$1 \times 10^5 + 5 \times 10^4 + 6 \times 10^3 + 7 \times 10^2 + 8 \times 10 + 2.$$

Now the result stated above for the product, the remainder when 10^2 is divided by 9 is the same as when the product of the remainders 1×1 is divided by 9, that is, the remainder is 1. Similarly 10^3 has a remainder $1 \times 1 \times 1$ when divided by 9 and hence 1. So all the powers of ten have a remainder 1 when divided by 9. Thus, by the result stated above for the sum, the remainder when 156,782 is divided by 9 is the same as the remainder when $1 \times 1 + 5 \times 1 + 6 \times 1 + 7 \times 1 + 8 \times 1 + 2$ is divided by 9. This last is just the sum of the digits. Writing it this way it is easy to see that this works for any number.

Now we can use the result of Exercise 13 to describe a check called "casting out the nines" which is not used much in these days of computing machines, but which is still interesting. Consider the product 867×934. We indicate the following calculations:

867 sum of digits: 21	sum of digits: 3
934 sum of digits: 16	sum of digits: 7
Product: 809,778	Product: $3 \times 7 = 21$
Sum of digits: $8 + 0 + 9 + 7 + 7 + 8 = 39$	
Sum of digits: $3 + 9 = 12$	
Sum of digits: $1 + 2 = 3$	Sum of digits: $2 + 1 = 3$.

Since the two results 3 are the same, we have at least some check on the accuracy of the results.

Exercises 3-3

1. Try the method of checking for another product. Would it also work for a sum? If so, try it also.

2. Explain why this should come out as it does.

3. If a computation checks this way, show that it still could be wrong. That is, in the example given above, what would be an incorrect product that would still check?

4. Given the number $5 \cdot 7^5 + 3 \cdot 7^4 + 2 \cdot 7^3 + 1 \cdot 7^2 + 4 \cdot 7 + 3$. What is its remainder when it is divided by 7? What is its remainder when it is divided by 6? by 3?

5. Can you find any short-cuts in the example above analogous to casting out the nines?

6. In a numeration system to the base 7 what would be the result corresponding to that in the decimal system which gives casting out the nines?

7. The following is a trick based on casting out the nines. Can you see how it works? You ask someone to pick a number -- it might be 1678. Then you ask him to form another number from the same digits in a different order -- he might take 6187. Then you ask him to subtract the smaller from the larger and give you the sum of all but one of the digits in the result. (He would have 4509 and might add the last three to give you 14.) All of this would be done without your seeing any of the figuring. Then you would tell him that the other digit

in the result is 4. Does the trick always work?

One method of shortening the computation for a test by casting out the nines, is to discard any partial sums which are 9 or a multiple of 9. For instance, in the example given, we did not need to add all the digits in 810,645. We could notice that $8 + 1 = 9$ and $4 + 5 = 9$ and hence the remainder when the sum of the digits is divided by 9 would be $0 + 6$, which is 6. Are there other places in the check where work could have been shortened? We thus, in a way, throw away the nines. It was from this that the name "casting out the nines" came.

By just the same principle, in a number system to the base 7 one would cast out the sixes, to the base 12 cast out the elevens, etc.

3-4. Divisibility by 11

There is a test for divisibility by 11 which is not quite so simple as that for divisibility by 9 but is quite easy to apply. In fact, there are two tests. We shall start you on one and let you discover the other for yourself. Suppose we wish to test the number 17945 for divisibility by 11. Then we can write it as before

$$1\cdot 10^4 + 7\cdot 10^3 + 9\cdot 10^2 + 4\cdot 10^1 + 5.$$

The remainders when 10^2 and 10^4 are divided by 11 are 1. But the remainders when 10, 10^3, 10^5 are divided by 11 are 10. Now 10 is equal to $11 - 1$. $10^3 = 10^2\,(11 - 1)$, $10^5 = 10^4\,(11 - 1)$. That is enough. Perhaps we have told you too much already. It is your turn to carry the ball.

Exercises 3-4-a

1. Without considering 10 to be 11 - 1, can you from the above devise a test for divisibility by 11?

2. Noticing that 10 = 11 - 1 and so forth as above, can you devise another test for divisibility by 11?

We hope you were able to devise the two tests suggested in the previous exercises. For the first, we could group the digits and write the number 17945 as $1 \times 10^4 + 79 \times 10^2 + 45$. Hence the remainder when the number 17945 is divided by 11 should be the same as the reaminder when $1 + 79 + 45$ is divided by 11, that is, $1 + 2 + 1 = 4$. (2 is the remainder when 79 is divided by 11, etc.) This method would hold for any number.

The second method requires a little knowledge of negative numbers (either review them or, if you have not had them, omit this paragraph). We could consider -1 as the remainder when 10 is divided by 11. Then the original number would have the same remainder as the remainder when $1 + 7(-1)^3 + 9 + 4(-1) + 5$ is divided by 11, that is, when $5 - 4 + 9 - 7 + 1$ is divided by 11. This last sum is equal to 4 which was what we got the other way. By this test we start at the right and alternately add and subtract digits. This is simpler than the other one.

Exercises 3-4-b

1. Test several numbers for divisibility by 11 using the two methods described above. Where the numbers are not divisible, find the remainders by the method given.

2. In a number system to the base 7, what number could we test for divisibility in the same way that we tested for 11 in the decimal system? Would both methods given above work for base 7 as well?

3. To test for divisibility by 11 we grouped the digits in pairs. What number or numbers could we test for divisibility by grouping the digits in triples? For example we might consider the number 157892. We could form the sum of 157 and 892. For what numbers would the remainders be the same?

4. Answer the questions raised in Exercise 3 in a numeral system to base 7 as well as in a numeral system to base 12.

5. In the repeating decimal for 1/9 in the decimal system there is one digit in the repeating portion; in the repeating decimal for 1/11 in the decimal system, there are two digits in the repeating portion. Is there any connection between these facts and the tests for divisibility for 9 and 11? What would be the connection between repeating decimals and the questions raised in Exercise 3 above?

6. Could one have a check in which 11's were "cast out?"

7. Can you find a trick for 11 similar to that in Exercise 1 above?

3-5. Divisibility by 7

There is not a very good test for divisibility by 7 in the decimal system. (In a numeration system to what base would there be a good test?) But it is worth looking into since we can see the connection between tests for divisibility and the repeating decimals. Consider the remainders when the powers of 10 are divided by 7. We put them in a little table:

n	1	2	3	4	5	6	7
Remainder when 10^n is divided by 7	3	2	6	4	5	1	3

If you compute the decimal equivalent for 1/7 you will see that the remainders are exactly the numbers in the second line of the table in the order given. Why is this so? This means that if we wanted to find the remainder when 7984532 is divided by 7 we would write

$$7 \times 10^6 + 9 \times 10^5 + 8 \times 10^4 + 4 \times 10^3 +$$
$$5 \times 10^2 + 3 \times 10 + 2$$

and replace the various powers of 10 by their remainders in the table to get

$$7 \times 1 + 9 \times 5 + 8 \times 4 + 4 \times 6 + 5 \times 2 + 3 \times 3 + 2.$$

We would have to compute this, divide by 7 and find the remainder. That would be as much work as dividing by 7 in the first place. So this is not a practical test but it does show the relationship between the repeating decimal and the test.

Notice that the sixth power of 10 has a remainder of 1 when it is divided by 7. If instead of 7 some other number is taken which has neither 2 nor 5 as a factor, 1 will be the remainder when some power of 10 is divided by that number. For instance, there is some power of 10 which has the remainder of 1 when it is divided by 23. This is very closely connected with the fact that the remainders must from a certain point on, repeat. Another way of expressing this result is that one can form a number completely of 9's, like 99999999, which is divisible by 23.

Exercise 3-5

Complete the following table. In doing this notice that it is not necessary to divide 10^{10} by 17 to get the remainder when it is divided by 17. We can compute each entry from the one above, like this: 10 is the remainder when 10 is divided by 17; this is the first entry. Then divide 10^2, that is, 100 by 17 and see that the remainder is 15. But we do not need to divide 1000 by 17. We merely notice that 1000 is 100 × 10 and hence the remainder when 1000 is divided by 17 is the same as the remainder when 15 × 10, or 150 is divided by 17. This remainder is 14. To find the remainder when 10^4 is divided by 17, notice that 10^4 is equal to $10^3 \times 10$ and hence the remainder when divided by 17 is the same as when 14 × 10 is divided by 17, that is 4. The table then gives the remainders when the powers of 10 are divided by various numbers.

	3	7	9	11	13	17	19	21	37	101	41
1	1	1	1			1					
10^1	1	3	1			10					
10^2	1	2	1			15					
10^3	1	6	1			14					
10^4	1	4	1			4					
10^5	1	5	1			6					
10^6	1	1	1			9					
10^7	1		1			5					
10^8	1		1			16					
10^9	1		1			7					
10^{10}	1		1			2					
10^{11}	1		1			3					
10^{12}	1		1			13					
10^{13}	1		1			11					
10^{14}	1		1			8					
10^{15}	1		1			12					
10^{16}	1		1			1					

Find what relationships you can between the number of digits in the repeating decimals for 1/3, 1/7, 1/9, 1/11, 1/13, etc. and the pattern of the remainders. Why does the table show that there will be five digits in the repeating portion of the decimal for 1/41? Will there be some other fraction 1/? which will have a repeating decimal with five digits in the repeating portion? How would you find a fraction 1/? which would have six digits in the repeating portion?

If you wish to explore these things further and find that you need help, you might begin to read some book on the theory of numbers. Also there is quite a little material on tests for divisibility in "Mathematical Excursions" by Miss Helen Abbott Merrill, Dover (1958).

SUPPLEMENTARY UNIT 4

GRAPHS: OPEN AND CLOSED PATHS*

4-1. The Seven Bridges of Königsberg

Here is a sketch of the map of the German city of Königsberg (now called Kaliningrad). Find out why.

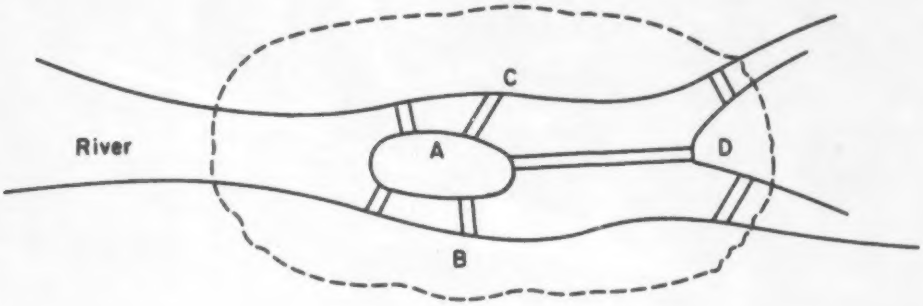


Figure 1

As you can see, a river goes through the city and divides into two branches on the east side, and there is an island in the middle of the river. There are seven bridges connecting the island and the different parts of the mainland.

After the great Swiss mathematician Euler (When did he live?) became court mathematician to Frederick the Great (Who was he? Why was he important?), a delegation came to him with a question,

---

*In this chapter we shall use the word "graph" with a meaning different from that in Unit 1, *Mathematics for Junior High School*, *Volume II*.

"Can a person go through the city of Königsberg in such a way as to pass over each bridge exactly once? We have worked for years on this problem and have never found an answer." (This problem was mentioned in Chapter 1.)

Euler replied, "Let me think. The exact shape of the different parts of the city doesn't matter. It would be simpler to represent them by points and the bridges by lines:

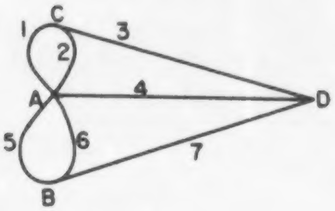


Figure 2

Of course, we only need to show how the parts are connected by the bridges. We can label the bridges, say, 1, 2, 3, 4, 5, 6, and 7. We could describe a path by a sequence as

A 1 C 3 D 4 A 6 B 5 2 C

showing the points in the diagram and the bridges over which the path passes, in the proper order. For example, this sequence stands for a path which starts at A, passes over bridge 1 to C, then goes over bridge 3 to D, and so on.

The question is, "Can we write such a sequence of letters and numbers in which each number appears just once?"

The men from Königsberg were amazed. Their jaws dropped in surprise. "Of course," they exclaimed. "It is really very simple, now that you explain it. If we had only thought of looking at the problem in this way, we could have solved it ourselves." They went home and tried to finish the problem.

That evening they sat around a table discussing the problem. One of them said, "Let us try some simple case first, just to get the hang of it. In this diagram

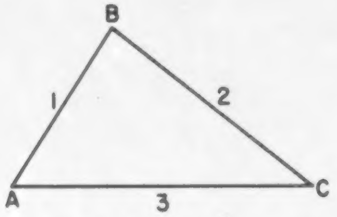


Figure 3

the path A 1 B 2 C 3 A goes over every bridge just once. It is a closed path because it comes back to the starting point."

Another man said, "I can't find a good path in this diagram:

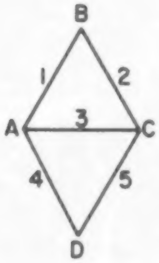


Figure 4

Is this one impossible, do you think?"

A third man remarked, "You could take the path

A 1 B 2 C 3 A 4 D 5 C.

It is an open path because it doesn't come back to the starting point. In this diagram there does not seem to be a closed path which goes over each bridge exactly once."

In this diagram

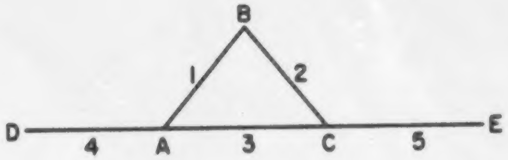


Figure 5

they could find neither a closed nor an open path. They worked hard until way past midnight, and still could not solve their problem.

The next day they came back to Euler and said, "We have been thinking about the problem, but still can't seem to solve it. There must be some simple idea which we have overlooked. If you could just get us started on the right track, we are sure that we can solve it ourselves."

Euler replied, "All right, let us look at Figure 4 where there is a path which goes over each bridge once and only once. The path is described by a sequence of letters, for the points, and numbers, for the bridges. Each number appears just once in this sequence because the path crosses each bridge just once."

"Sure enough," they said, "in the sequence

A 1 B 2 C 3 A 4 D 5 C

each number appears just once. The same is true of the sequence

A 1 B 2 C 3 A

in Figure 3."

Euler said, "Look at these sequences more carefully. What comes before each letter except the first?"

"A number," they answered. "This corresponds to a bridge leading to the point."

"What comes after each letter except the last?"

"A number, of course. There is also a bridge leading away from the point."

"How many bridges are there for each time the path goes through a point?"

"Two bridges. We come into the point on one bridge and go away from it on another bridge. For each time a letter appears in the path, except at the beginning or end, there are two numbers for these two bridges."

Euler suggested, "Let us call all points of the path, except for the two endpoints, inner points. Then for each inner point of the path there are two bridges. Suppose the point B appears three times as an inner point of the path. For instance, look at this diagram

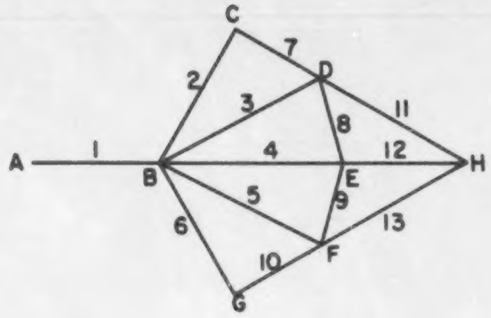


Figure 6

and the path A 1 B 2 C 7 D 3 B 4 E 9 F 5 B 6 G 10 F 13 H 12 E 8 D 11 H. How many bridges are connected to B?"

"Six," answered the men from Königsberg.

"How did you get that?" asked Euler.

"We simply multiplied the number of times the point appears by 2, the number of bridges connected with the point at each appearance."

"Will this always work?" Euler continued.

"Yes, for every inner point of the path."

"What kind of number do you get when you multiply some number by 2?" Euler asked again.

"Obviously, an even number." The men from Königsberg looked at each other, pleasantly surprised. "Then the total number of bridges leading to or from any inner point of the path must be even. Any child could see that!"

"What about the endpoints, the first and the last point?"

They thought for a moment. "Let us see. There is a bridge leading from the first point. Then every other time the path goes through this point, there are two bridges. So the total number of bridges connected to the first point is one more than an even number. In other words, it is an odd number. The same is true of the last point."

Euler questioned them further. "Are you sure? Must the first point be different from the last point?"

They smiled. "Of course not. Thanks for reminding us not to overlook that possibility. If the path is closed, that is, if it comes back to the starting point, then that point will be like any inner point of the path. Then the number of bridges to or from that point must be even."

Euler suggested, "It might be a good idea to summarize what you have figured out so far."

They said, "All right. If the path is closed, then there is an even number of bridges connected to each point. If the path is open, then each of the two endpoints must have an odd number of bridges. Each of the inner points is connected to an even number of bridges. Now that we think of it, the problem is absurdly simple."

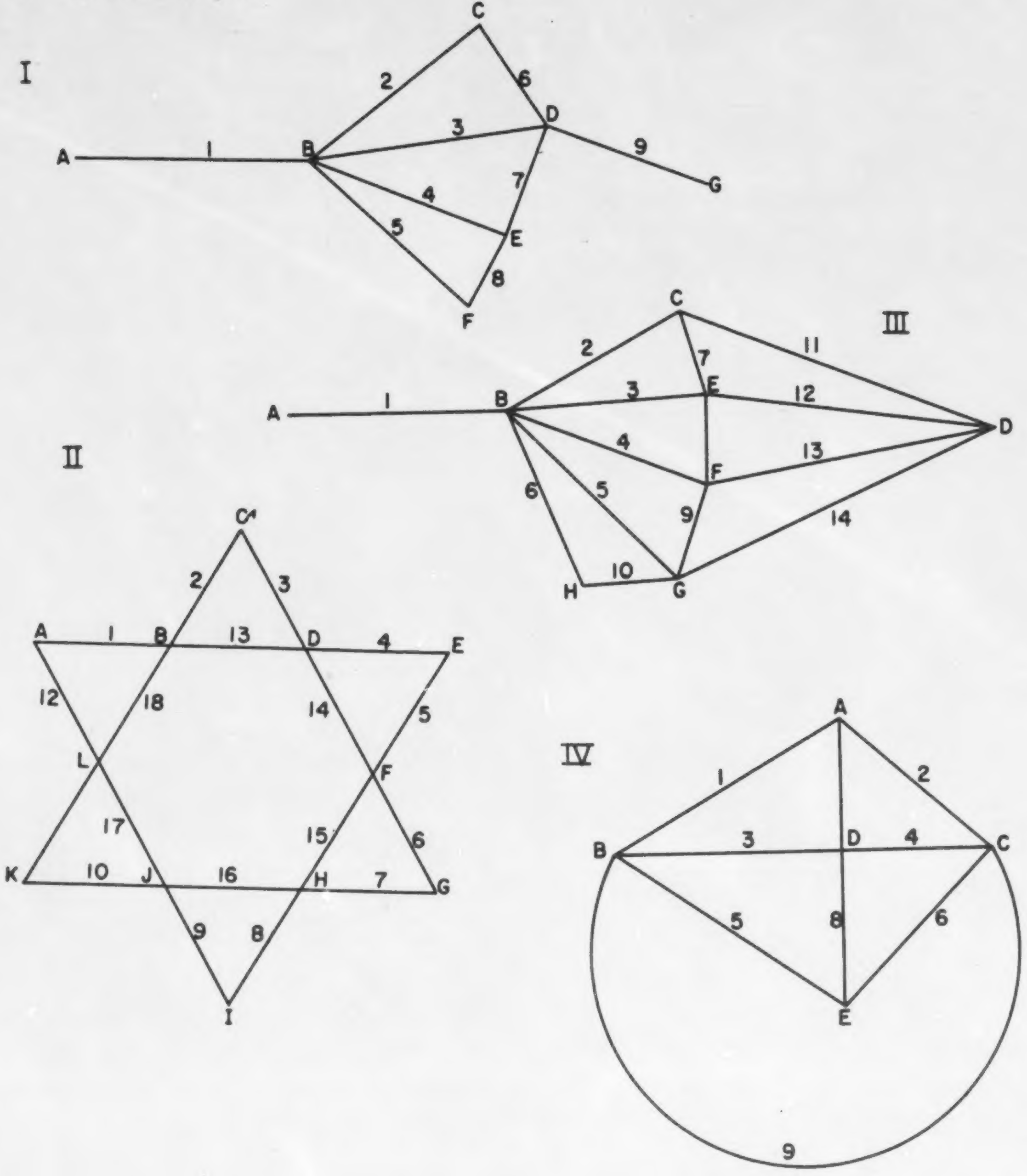
The men from Königsberg bent over the diagram and began counting. "The point C is connected to bridges 1, 2, and 3, the point D to bridges __________, the point A to bridges __________, and the point B to bridges __________. There are _____ points connected to an odd number of bridges and _____ points connected to an even number of bridges. Is a closed path possible? _____ (Yes, or no?) Is an open path possible? _____ (Yes, or no?) Such an easy problem, after all!" (Fill in the blanks yourself.)

After thanking Euler, the merry gentlemen from Königsberg went home. On the way, one of them said, "I don't see why Euler has such a great reputation. We really worked out every step of the problem ourselves. All Euler did was to suggest how to look at the problem and ask the right questions." His companions nodded and replied, "Yes, the problem was really so elementary that any child could have solved it."

What do you think?

Exercises 4-1

Here are some diagrams with some points connected by bridges in various ways.


I
A
B
C
D
E
F
G
1
2
3
4
5
6
7
8
9
III
A
B
C
D
E
F
G
H
1
2
3
4
5
6
7
9
10
11
12
13
14
II
A
B
C
D
E
F
G
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I
J
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3
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5
6
7
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9
10
12
13
14
15
16
17
18
IV
A
B
C
D
E
1
2
3
4
5
6
8
9

Figure 7

1. (a) For each diagram list the points which are connected to an even number of bridges.
   (b) List the points connected with an odd number of bridges.
   (c) How many points of each kind are there in each diagram?

2. (a) In which diagrams is it impossible to find a closed path which goes over every bridge just once?
   (b) In which diagrams is it impossible to find an open path of this kind?

3. For each of the diagrams where it might be possible to have a path going over each bridge exactly once, look for such a path. If you do find a path, describe it by a sequence of letters and numbers.

4. For each of these diagrams find a closed path starting at the point B which goes over each bridge just once, and which goes over the largest possible number of bridges.

5. In Figure 4 there are three other paths from A to C which go over each bridge exactly once. One of them is described by the sequence A 4 D 5 C 2 B 1 A 3 C. Find the other two.

4-2. What Happens if There Is a Path

A set of points and bridges, in which each point has at least one bridge attached to it, we call a graph. The points are called vertices (singular: vertex) of the graph. A vertex is called even if an even number of bridges are connected to it. Otherwise the vertex is called odd. A path is called closed if its last vertex is the same as its first vertex. Otherwise the path is called open. Notice that we are using the word "graph" in a special way in this chapter. Don't confuse this meaning with the meaning in Unit 1. Compare footnote bottom of page 67.

By using the same reasoning that the men from Königsberg used, with Euler's help, you can prove the general statements:

Theorem 1. If there is, in a graph, a closed path which goes over each bridge just once, then every vertex is even. If there is an open path of this kind, then there are two odd vertices, and all the rest are even.

(A theorem is a statement proved by logical reasoning.)

Exercises 4-2

1. In the graphs of Exercises 1, name the odd and the even vertices. How many odd vertices are there in each graph? Does there seem to be a general principle?

2. State a general principle about the number of odd vertices in any graph which seems to be true in all cases. Draw five more graphs, and test whether your statement is true in each case. Compare your results with those of your classmates.

In any graph you may classify the vertices more precisely according to the number of bridges connected with each one. The number of bridges leading to or from a vertex we shall call the degree of the vertex. In Figure 2 vertex A is of the 5th degree, whereas the others are of degree 3.

3. For each of the graphs you have drawn, make a table showing the number of vertices of each degree, like this:

degree	number of vertices
1	
2	
3	
4	
5	
etc.	

How is the total number of vertices related to the numbers in the right hand column?

4. Call the total number of vertices in a graph V. Let V_1 be the number of vertices of degree 1, V_2 the number of degree 2, etc. (The numbers $V_1, V_2, \cdots$, are the numbers in the right hand column in the above table.) Express the relation between V and the numbers V_1, V_2, etc. as an equation.

5. Take any graph. Label the bridges with numbers and the vertices with letters. List all pairs consisting of a vertex and a bridge connected to it. In Figure 2 the pairs are named:

   A1, A2, A4, A5, A6, B5, B7, C1, C2, C3, D3, D4, D5.

6. In Exercise 5, in how many pairs does a given bridge occur? How is the number of pairs related to the number of bridges? Let B be the number of bridges. Give a formula for the

number of pairs in terms of B.

7. In Exercise 5, in how many pairs does a given vertex of degree 3 occur? In how many pairs does a given vertex of degree k occur? What is the total number of pairs in which a vertex of degree 3 occurs? What is the total number of pairs in which a vertex of degree k occurs?

8. Give a formula for the total number of pairs in Exercise 5 in terms of the numbers V_1, V_2, V_3, $\cdots$.

9. Give a formula for the total number of odd vertices in terms of V_1, V_2, V_3, $\cdots$.

10. Let U be the total number of odd vertices. Give a formula for the number $(2 \cdot B) - U$ in terms of V_1, V_2, V_3, etc.

11. Can you use the formula in Exercise 10 to prove the principle you discovered in Exercise 2?

4-3. When Can You Be Sure That There is a Path?

According to Theorem 1, if there is a closed path in a graph which goes over each bridge exactly once, then a certain thing is true. This is a necessary condition that there be such a path in a graph. If a graph does not satisfy this condition, namely that all its vertices are even, then we are sure that there is no closed path going over each bridge just once.

Is this condition sufficient? If all the vertices are even does there exist a path of this kind in the graph? Examine all the graphs you have drawn so far. Find the ones which have only even vertices. Can you find, in each one of these a closed path going over each bridge once and only once? Can you draw a counterexample, a graph with only even vertices in which there is no such path?

Does it seem as though the condition that the graph have no odd vertices is sufficient? Compare your conclusions with those of your classmates before you turn this page.

Look at this graph:

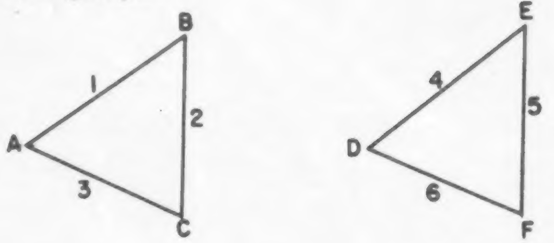


Figure 8

Are there any odd vertices? Can you find a path which goes over every bridge just once? In fact, is there any path which goes over both bridges 1 and 4? If you are not sure whether this is a graph, reread the definition of a graph. This will teach you why we must be so careful in mathematics to say exactly what we mean.

The trouble with Figure 4 is that it is made up of two separate pieces. There is no use looking for a path which goes over every bridge unless the graph is connected. We say that a graph is connected if every vertex can be joined to any other vertex by a path. In Figure 4 the vertex A can be joined to B and C, but not to any of the other vertices.

It turns out that if a connected graph has no odd vertices, then there is a closed path which goes over every bridge exactly once. We shall lead you to discover the proof in two stages.

Theorem 2. If a graph has no odd vertices, then through every vertex there is a closed path which doesn't go over any bridge twice.

Proof: Suppose Q_1 is a vertex of the graph. Find the longest path (measured by the number of bridges in it) which starts at Q_1 and doesn't go over any bridge more than once. Suppose, for example, that this path has 7 bridges in it. We could describe the path roughly like this:

$$Q_1Q_2Q_3Q_4Q_5Q_6Q_7Q_8.$$

Here the subscripts simply help us name the vertices. For example, Q_2 is the second vertex. We did not bother to write the numbers of the bridges between the names of the vertices. Now suppose Q_8 is not the same as Q_1. Is this path open or closed? Is Q_8 an inner point or an endpoint of this path? What do you know about the number of bridges connected to an endpoint of a path? What was assumed about the total number of bridges connected to any point of the graph? Can this path contain all the bridges connected to Q_8?

If not, then there is at least one more bridge in the graph, connected to Q_8 but not in this path. If we go over this bridge, too, then we will have a path

$$Q_1Q_2Q_3Q_4Q_5Q_6Q_7Q_8Q_9$$

starting at Q_1 with 8 bridges. This contradicts our assumption that the longest path, starting at Q_1, in the graph has only 7 bridges.

Since we got into a contradiction by assuming that Q_8 was not the same as Q_1, then this assumption must be false. Therefore, Q_8 is the same as Q_1, so this is a closed path through

Q_1 which doesn't go over any bridge twice.

Now you are ready for the second stage:

Theorem 3. If a connected graph has only even vertices, then there is a closed path going over every bridge just once.

Proof: Suppose you look at the longest such path in the graph. Color the bridges and vertices of this path blue. If this path does not contain every bridge, then color in red all bridges which are not in this path. We are going to assume that there is a red bridge, and see what follows. We claim that there is a purple vertex, that is one colored both blue and red.

To see this, take any red bridge and some blue vertex P. Since the graph is connected, there is a path joining either vertex, say Q_1 of the given red bridge with the vertex P. Look at the last red bridge in this path. Suppose it leads from the vertex R to the vertex S. Since this bridge is red, then S is colored red. But the next bridge in the path is blue. Therefore, S is also blue. So S is purple.

Now look at the graph made up of the red bridges, which we can call simply the red graph. Since the blue path is closed, there is an even number of blue bridges connected to each of its vertices. Since the total number of bridges connected to any vertex of the original graph is even, that leaves an even number of red bridges (possibly 0) connected to any vertex.

Therefore in the red graph, there is an even number of bridges connected to each vertex. We can apply Theorem 2 to the red graph. Hence there is a closed path in the red graph through the purple vertex S. We have then a picture like this:

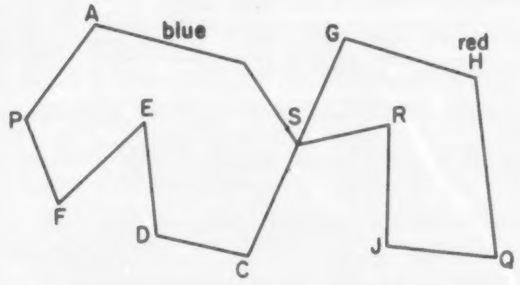


Figure 9

Then the path P A B S G H Q J R S C D E F P is a closed path which doesn't go over any bridge more than once. This path is longer than the blue path. This is a contradiction since the blue path was supposed to be the longest such closed path in the graph.

We got into trouble by assuming that the blue path did not contain all the bridges. Therefore, it does contain all of them. So the blue path is the one we were looking for.

Exercise 4-3

BRAINBUSTER: Prove that if a connected graph has 2 odd vertices and all the rest even, then there is an open path which goes over every bridge exactly once.

4-4. Hamiltonian Paths

There is another problem which sounds no more difficult than Euler's problem. Yet no one knows the answer. Because the first problem of this kind was solved by the great Irish mathematician, Sir William Rowan Hamilton (When and where did he live?), the paths we seek are named after him.

A Hamiltonian path is a graph in a closed path which goes through every vertex of the graph without going over any bridge more than once. A Hamiltonian path does not have to go over every bridge in the graph. Figure 10 shows a graph with a Hamiltonian path:

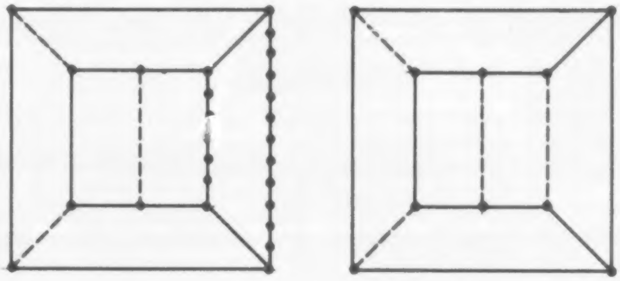


Figure 10

The dotted lines represent bridges which are not in the Hamiltonian path.

Exercise 4-4

In which of the following graphs is there is Hamiltonian path?

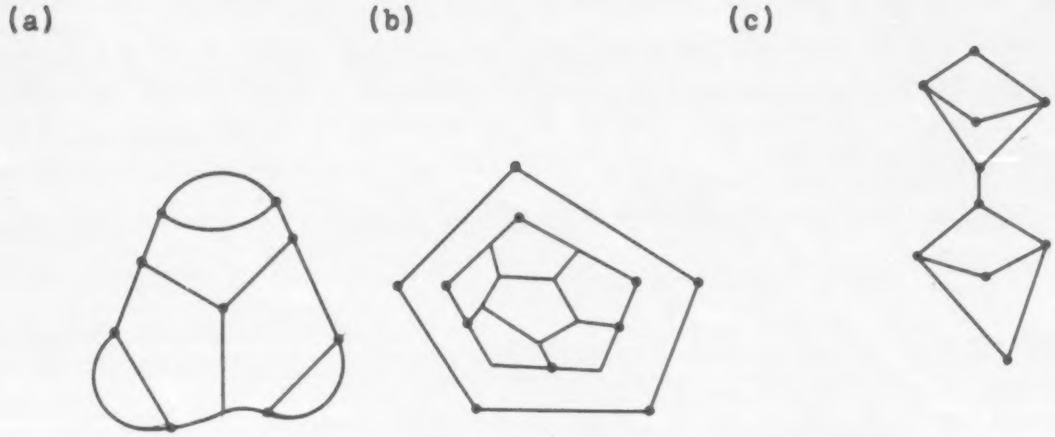


Figure 11

A necessary and sufficient condition for a graph to contain a Hamiltonian path is unknown. This is one way for you to become world famous overnight. Good luck to your efforts! We hope you have lots of fun trying.

SUPPLEMENTARY UNIT 5

FINITE DIFFERENCES

5-1. Arithmetic Progressions

Suppose we look at a few interesting sets of numbers to begin with, and take differences of successive numbers:

Table I

1		2		3		4		5		6	...	n		(n 1)	...
	1		1		1		1		1		...		1		...

Between each pair of numbers and on the line below it we write the difference:

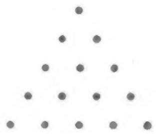
$$2 - 1 = 1,\ 3 - 2 = 1,\ 4 - 3 = 1,\ \ldots\ .$$

It begins to be monotonous after a while. Why did we have the number n? It was just to indicate any number (n stands for "any"). The next number after n would be $(n + 1)$ since in this "sequence" you get each number by adding 1 to the number before. (When we have a set of numbers in some order, we call it a "sequence.") What would be the next one after $(n + 1)$? What would be the one before n? You should read this unit with a pencil and sheet of paper at hand so that you may answer these questions as they occur. You may also have questions of your own which you would like to try to answer.

There is nothing especially strange about the differences being 1's since each time you added 1 to get the next entry.

Could you write a sequence in which all the differences are 2's or 3's or any other number? Any sequence for which the difference between successive numbers is the same every time is called an arithmetic progression.

Let us look back to the numbers of Table I. There is a connection with the game of ten pins or bowling. Look at the triangle of dots below:



If we omitted the last line we would have the usual arrangement of ten pins in a bowling alley. If there were just one row we would have the number 1, if two rows the number 3, if three rows the number 6, etc. These are called "triangular numbers." We write these in

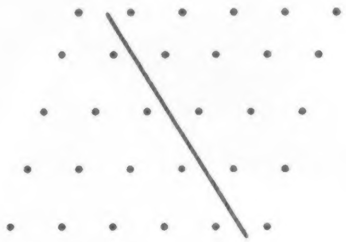
Table II

	1		3		6		10		15		21		28		...
Differences		2		3		4		5		6		7		...	

If we compare this table with Table I we can notice a number of interesting things. The first entries in the two tables are each 1. The second entry in Table II is the sum of the first two entries in Table I, the third entry in Table II is the sum of the first three entries in Table I, etc. The tenth entry in Table II would be the sum of the first ten entries in Table I. We could also say that the n-th entry in Table II (we do not yet have a formula for it) is the sum of the first n entries in Table I.

Another thing we notice in comparing the two tables is that the differences in the second line of Table II are the same as the entries in the first line of Table I except for the first one. Why is this so? Of course if we had written in Table II a third line giving the differences for the second line we would have had a succession of 1's as before.

Now we could find the sum of the first ten numbers in Table I by adding them - this would give us the tenth entry in the first line of Table II, but this would be rather tedious. There is an interesting little trick that will give us our result with less effort. Suppose we form another triangle of dots like that above, turn it upside down and fit it carefully next to the one already written. Then we would have a figure like:



In this picture we have 5 rows with 6 dots in each row, which gives $5 \times 6 = 30$ dots in all. Hence the number of dots in the first triangle would be $1/2 \times 30 = 15$, which is the fifth triangular number. If we wanted the 20th triangular number we would have a triangle of 20 rows. If we make another triangle of dots and place it as we did for the smaller triangle, we would have 20 rows with 21 dots each and hence 20×21 dots in the two triangles together, which implies that in each triangle there would be

$$1/2 \times 20 \times 21$$

dots. So the 20th triangular number is 210, which is the same as the sum of the numbers 1, 2, 3, ··· up to and including 20.

By this means we would find in the same manner the number of dots in any triangular array of this kind, that is, we could find any triangular number. Let us write a few:

40th triangular number: $1/2 \times 40 \times 41 = 820$

100th triangular number: $1/2 \times 100 \times 101 = 5050$

120th triangular number: $1/2 \times 120 \times 121 = 7260$.

In each case we would take the product of 1/2, the number and 1 more than the number. We could get a formula by letting n stand for the number and say that

the n-th triangular number: $1/2 \times n \times (n+1)$.

Then we would get the above three values by letting $n = 40$, $n = 100$, $n = 120$. Any triangular number we could get by using

$$1/2\, n\, (n + 1)$$

where this is another way of writing $1/2 \times n \times (n + 1)$.

We could also get this result without any reference to dots by use of a trick that is suggested by the triangles we drew. Suppose we wanted the 20th triangular number. Then we could take the sum in two different orders:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & \ldots 17 & 18 & 19 & 20 \\ 20 & 19 & 18 & 17 & 4 & 3 & 2 & 1. \end{array}$$

The sum of each column is 21, there are 20 columns and hence the sum of the numbers in the two rows is 20 × 21 and that in each row is one-half of this. We could do this for any number in place of 20 and one way of showing this would be to write it out using n for the number in place of 20 or whatever number we had. It would look like this:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \ldots & (n-1) & n \\ n & n-1 & n-2 & n-3 & \ldots & 2 & 1. \end{array}$$

The sum of each column is $n + 1$ and there are n columns. Hence the **sum of** all the numbers in the two rows is $n(n + 1)$ and half this is the sum for each row.

We shall find still another way to get this sum in the next section.

Exercises 5-1

1. Write another sequence of numbers for which the differences are all 1's. What would be the sum of the first 20 numbers? Can you give a formula for the sum of the first n numbers?

2. Write a sequence of numbers for which the differences are all 2's. What would be the sum of the first 20 numbers? Can you give a formula for the sum of the first n numbers?

3. Consider the formula: $2n + 7$ (remember that $2n$ means $2 \times n$). When $n = 1$, $2n + 7$ is $2 \times 1 + 7 = 9$; when $n = 2$, $2n + 7$ is $2 \times 2 + 7 = 11$, etc. We can form a table of values:

n	1	2	3	4	5	6
$2n + 7$	9	11	13	15	17	19

Carry this table out for the next three values of n. Use the numbers 9, 11, 13, $\cdots$ as the first row of a table and then write below this row a row of differences. Do you notice any relationship between the formula and these differences?

4. Do the same as in problem 3 for the formula $3n + 7$ and for $2n + 6$.

5. What would be the differences for the numbers defined by the formula $5n + 7$?

6. Write the first 20 odd numbers. Can you find their sum without just adding them? Can you guess what a formula for the sum of the first n odd numbers would be? Try to prove your result.

7. Give a formula for the sum of the first $n - 1$ numbers in Table I.

8. Find a formula for the sum of the following:

$$1,\ 1 + d,\ 1 + 2d,\ \cdots,\ 1 + nd.$$

9. Give a formula for the sum of the following:

$$1,\ 1 + d,\ 1 + 2d,\ \cdots,\ 1 + (n - 1)d.$$

10. Find a formula for the sum of the same sequence of the previous problem except that 1 is replaced by b.

11. Suppose the first two numbers in a table are

7 and 12.

Write a table starting with these two numbers for which the first differences are all the same, that is, in which the numbers on the first row are in an arithmetic progression.

12. Write a table of numbers in an arithmetic progression in which the first two entires are 7 and 5 in that order.

13. If you have any two numbers instead of 7 and 12, or 7 and 5, could you make a table starting with the two given numbers in which the numbers of the first row form an arithmetic progression? Give reasons.

5-2. More Sequences

Now form a table of the squares of the integers. Recall that the square of 3 is 9 since $3 \times 3 = 9$, the square of 5 is 25 since $5 \times 5 = 5^2 = 25$, etc. We call them "squares" or "square numbers" because if we wrote our dots in squares instead of triangles, as previously we would have the following sequence of squares:

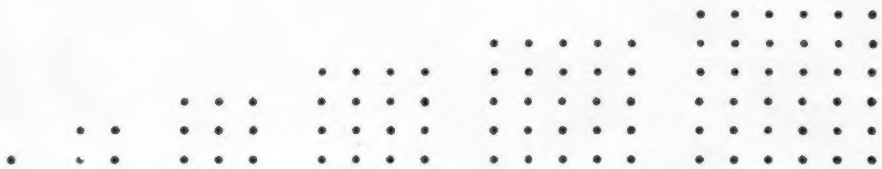


Table III

1 4 9 16 25 36 49 ... n^2 $(n + 1)^2$

3 5 7 9 11 13 ... w

2 2 2 2 2 ... 2

Notice that the numbers here in the second row are in an arithmetic progression and that the differences in the third row are all 2's. We call the numbers in the second row of such a table "first differences" and those in the third row "second differences." What would be the n-th term in the second row, that is, the entry where w is? (w stands for "what.") This should not be hard to find since it is the difference of the two numbers above it. It is just

$$(n + 1)^2 - n^2.$$

Before getting a simpler expression for this difference of two squares, let us see how it goes for some of the numbers. Just to write $36 - 25 = 11$ is not especially enlightening. But suppose we write it as

$$6^2 - 5^2 = (5 + 1)^2 - 5^2.$$

If we use the distributive property several times we have:

$$\begin{aligned}(5 + 1)^2 &= (5 + 1) \times (5 + 1) = 6 \times (5 + 1)\\ &= 6 \times 5 + 6 \times 1 = (5 + 1) \times 5\\ &+ (5 + 1) \times 1 = 5^2 + 1 \times 5 + 5 \times 1\\ &+ 1 = 5^2 + 2 \times 5 + 1.\end{aligned}$$

And thus

$$6^2 - 5^2 = 5^2 + 2 \times 5 + 1 - 5^2 = 2 \times 5 + 1.$$

In just the same way we could show that

$$7^2 - 6^2 = 6^2 + 2 \times 6 + 1 - 6^2 = 2 \times 6 + 1.$$

(Try it and see.) So, putting n in place of 5 or 6 or whatever number, we have

$$(n + 1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1.$$

We could write this in words: The difference between the squares of two successive integers is 1 more than twice the smaller one. For instance: $121^2 - 120^2 = 2 \times 120 + 1 = 241$. This is a much simpler computation than squaring both numbers and taking the difference. This can also be shown using diagrams of squares in dots, but this is left as an exercise.

This shows that the last entry in the second row of Table III should be $2n + 1$. We might check this: when n is 1, $2n + 1$ is 3; when n is 2, $2n + 1$ is 5, etc.

The numbers in the second row are in an arithmetic progression. If you look carefully, you will see that each number in the first row is 1 more than the sum of the numbers to the left of it in the row below. Why is this so? Another way of saying this is that the fifth number in the first row is the sum of the first five odd numbers, the sixth number in the first row is the sum of the first six odd numbers, etc. What would be the sum of the first 20 odd numbers?

We can use this to get the formula for the sum of the first n numbers in still another way. Start with

$$(1) \quad 3 + 5 + 7 + \cdots + (2n + 1) = (n + 1)^2 - 1 = n^2 + 2n.$$

Notice that 3 is the value of $2n + 1$ when $n = 1$, 5 is the value of $2n + 1$ when $n = 2$, etc. Then we can write the left side of equation (1) as follows:

$$(2 \times 1 + 1) + (2 \times 2 + 1) + (2 \times 3 + 1) + \cdots + (2n + 1).$$

If we write this in a different order, using the commutative property, we have

$$2 \times 1 + 2 \times 2 + 2 \times 3 + \ldots + 2n + (1 + 1 + 1 + \cdots + 1)$$

where there are n 1's in the parentheses. Then, from the distributive property, this can be written

$$2 \times (1 + 2 + 3 + \cdots + n) + n.$$

If we put this in for the left side of equation (1) we get the equation:

$$2 \times (1 + 2 + 3 + \cdots + n) + n = n^2 + 2n.$$

Subtract n from both sides to get

$$2 \times (1 + 2 + 3 + \cdots + n) = n^2 + 2n - n = n^2 + n.$$

Finally, if we divide both sides by 2 we have

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}(n^2 + n) = \frac{1}{2}n(n + 1)$$

which is the formula we had before for the n-th triangular number.

This is, of course, a much harder way to find the sum of the first n integers than by the other methods. But it does give us a means of finding the sum of the squares; for just as we got the sum of the integers by considering the squares, we should be able to get the sum of the squares of the integers by considering a table of their cubes. Let us try it.

Table IV

1	8	27	64	125	216	...	n^3	$(n + 1)^3$ ...
7	19	37	61	91	...		w	
12	18	24	30	...				
6	6	6						

Notice that here it is the second differences which form an arithmetic progression and the third differences which are all the same.

The second row should be connected somehow with the squares of the integers. To get a clue for this connection, we must determine the formula for the last term in the second row, which we have called w. This is just

$$(n + 1)^3 - n^3.$$

To work this out, let c temporarily stand for $(n + 1)^2$ and have

$$(n + 1)^3 = (n + 1) \times (n + 1)^2 = (n + 1) \times c$$
$$= n \times c + 1 \times c = n \times c + c.$$

We found previously that $(n + 1)^2 = n^2 + 2n + 1$, and replacing c by this, we have

$$(n + 1)^3 = n \times (n^2 + 2n + 1) + n^2 + 2n + 1$$
$$= n^3 + 2n^2 + n + n^2 + 2n + 1$$
$$= n^3 + 3n^2 + 3n + 1.$$

Thus

$$(n + 1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1.$$

To check this, let us form a little table of values:

n	1	2	3	4
$3n^2 + 3n + 1$	7	19	37	61

which checks with the second row of Table IV.

From this we are now going to work out the following formula for the sum of the first n squares:

$$s = \frac{2n^3 + 3n^2 + n}{6}.$$

If you find the algebra too difficult, you can just assume the formula and go on to the exercises after checking the formula for a few values of n.

To get the formula first notice that in Table IV, $8 = 1 + 7$, $27 = 1 + 7 + 19$, $64 = 1 + 7 + 19 + 37$, etc. Each number in the first row after the 1 is 1 more than the sum of the numbers in the second row and to the left of it. That is, $(n + 1)^3$ is 1 plus the sum of the numbers in the second row through w, which is $3n^2 + 3n + 1$. Hence we have the following equation:

$$7 + 19 + 37 + \cdots + (3n^2 + 3n + 1) = (n + 1)^3 - 1. \tag{2}$$

From our work above we see that the right side of this equation is equal to

$$n^3 + 3n^2 + 3n + 1 - 1 = n^3 + 3n^2 + 3n,$$

and the left side may be written:

$$\begin{array}{l} (3 \times 1^2 + 3 \times 1 + 1) + \\ (3 \times 2^2 + 3 \times 2 + 1) + \\ (3 \times 3^2 + 3 \times 3 + 1) + \\ \cdots\cdots\cdots\cdots \\ (3 \times n^2 + 3 \times n + 1). \end{array}$$

Notice that the numbers after the first multiplication signs are the squares of the numbers from 1 to n, the numbers after the second multiplication signs are the numbers from 1 to n and the last number in each line is 1. So if we add by columns we have, using the distributive property:

$$\begin{array}{c} 3 \times (1^2 + 2^2 + 3^2 + \cdots + n^2) + \\ 3 \times (1 + 2 + 3 + \cdots + n) + \\ (1 + 1 + 1 + \cdots + 1), \end{array}$$

where in the last line there are n 1's. We have called s the sum of the squares of the first n integers, we know that the sum of the first n integers is $\frac{1}{2}(n^2 + n)$ and the sum of the n 1's is n. Hence the expression can be abbreviated to:

$$3s + 3 \times \frac{1}{2}(n^2 + n) + n,$$

which is what the left side of (2) reduces to. If we equate it to what we found above for the right side we have:

$$3s + 3 \times \frac{1}{2}(n^2 + n) + n = n^3 + 3n^2 + 3n.$$

Since:

$$n = \frac{2n}{2}, \quad 3 \times \frac{1}{2}(n^2 + n) = \frac{3n^2 + 3n}{2}$$

and

$$n^3 + 3n^2 + 3n = \frac{2n^3 + 6n^2 + 6n}{2}$$

our equation becomes:

$$3s + \frac{3n^2 + 3n + 2n}{2} = \frac{2n^3 + 6n^2 + 6n}{2}.$$

Notice that $3n + 2n = 5n$ and substract $\frac{3n^2 + 5n}{2}$ from both sides to get

$$3s = \frac{2n^3 + 6n^2 + 6n - 3n^2 - 5n}{2} = \frac{2n^3 + 3n^2 + n}{2}.$$

Finally if we divide both sides by 3 we have the formula

$$s = \frac{2n^3 + 3n^2 + n}{6}$$

which is what we stated above.

You should check this for the first two or three values of n.

Exercises 5-2

1. Using dots in the form of squares, show that

$$(n + 1)^2 - n^2 = 2n + 1.$$

2. Find a formula for the sum of the squares of the first n even integers. (You may want to make a table first.)

3. Find a formula for the sum of the squares of the first n odd integers. Hint: notice that $(2n - 1)^2 = 4n^2 - 4n + 1$.

4. Given the numbers 4, 7, 12, can you form a table beginning with these numbers in which the first differences are in an arithmetic progression?

5. Answer the same question as that in problem 4 but with the numbers 4, 7, 12 replaced by 10, 5, 11 in that order.

6. Given any three numbers, could a table be constructed having the given numbers as the first three entries in order and for which the first differences would be in an arithmetic progression? Give reasons for your answers.

7. Find a formula for the sum of the first n cubes of integers, that is, for 1, 8, 27, 64, etc.

5-3. Finding Formulas that Fit

By the methods we used in the previous sections we could find formulas for the sums of cubes, fourth powers, fifth powers and so on but the computations and algebra become more and more difficult. It is time we tried something else.

We can use some of the same methods to find formulas to fit some tables of values. Suppose we had the sequence of numbers:

$$3 \quad 7 \quad 11 \quad 15 \quad 19 \quad \cdots$$

and we wanted a formula that would fit these values. We could form a table and take the first differences

Table V

$$\begin{array}{ccccccccc} 3 & & 7 & & 11 & & 15 & & 19 \\ & 4 & & 4 & & 4 & & 4 & \end{array}$$

These differences are all the same, that is, the numbers in the first row are in an arithmetic progression. (Of course the next value might not be, but we are only trying to find a formula which fits the given values.) From this we might guess that the formula for the numbers in the first row would be of the form: $an + b$ for some numbers a and b. Suppose we try it to see if it works. Then the n-th and $(n + 1)$st entries would be

$$an + b \quad \text{and} \quad a(n + 1) + b$$

and their difference would be

$$a(n + 1) + b - an - b = an + a + b - an - b = a$$

which is the difference. Since all the differences are 4, it follows that a must be 4 and our formula becomes

$$4n + b.$$

Now when n is 1, $4n + b$ must be the first entry, that is

$$4 + b = 3$$

which means that b must be -1 and hence the formula seems to be

$$4n - 1.$$

If we try this for various values of n we see that it works and this indeed fits the five entries in the first row of the table.

Actually we could see that this would have to work if the numbers are in an arithmetic progression, once we have fixed b so that the first entry fits the formula; for, whatever b is, the numbers in the first row would be

$$b \qquad 4 + b \qquad 2 \times 4 + b \qquad 3 \times 4 + b$$

and the differences are all 4's.

Really we have proved more than we set out to do. We have the

Theorem: If the first differences of a table of values are all the same, call them a, then the numbers form an arithmetic progression and the formula for the n-th term is

$$an + b$$

where b is so chosen that $a + b$ is the first number in the table.

By means of this theorem we could get a formula to fit any table of values in an arithmetic progression, that is, in which the first differences are all equal. What about tables in which this is not the case? In order to explore this, suppose we test the tables for a few formulas to see if we can make some guesses.

Table for $n(n + 2) = n^2 + 2n$

n	1	2	3	4	5	...
$n(n + 2)$	3	8	15	24	35	...
first differences	5	7	9	11		

Here the first differences form an arithmetic progression. (You should check these values and compute a few more.)

Table for n(n + 1)(n + 2)

n	1		2		3		4		5		6 ···	
n(n + 1)(n + 2)	6		24		60		120		210		336 ···	
First differences		18		36		60		90		126 ···		
Second differences			18		24		30		36 ···			

Notice that n(n + 1)(n + 2) is the product of three successive integers beginning with n. Here it is the second differences which are in an arithmetic progression. This would give us a way of computing the values of n(n + 1)(n + 2) successively, assuming that the second differences are in an arithmetic progression no matter how far one goes in the table. For instance, the next second difference would be 42 = 36 + 6, the next first difference would be 126 + 42 = 168 which means that the next entry in the line above would be 336 + 168 = 504. To check this we see that 504 = 7 × 8 × 9. (Notice that every number after the first line in the table is divisible by 6. Why is this so?)

Try one more table:

Table for n(n + 1)(n + 2)(n + 3)

n	1		2		3		4		5		6
n(n++ 1)(n + 2)(n + 3)	24		120		360		840		1680		3024
First differences		96		240		480		840		1344	
Second differences			144		240		360		504		
Third differences				96		120		144			

Here it is the third differences that are in an arithmetic progression. Notice that every number after the first row is divisible by 24. Why is this so?

Before going further, you should try out a few for yourself.

Exercises 5-3-a

1. Find tables of values for each of the following formulas and compute first, second, third differences:

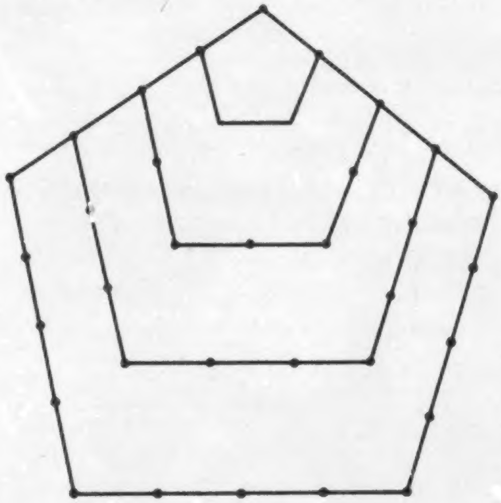
   (a) $n^2 + 3n + 2$

   (b) $\frac{n^3 - n}{6}$

   (c) $n^3 + n$

2. Suppose you computed a table for the formula: $n^4 - n^2$ and computed the first, second, etc. differences. Guess how soon you would come to an arithmetic progression. Then check it to find out.

Now we can come back to the problem of trying to find formulas that fit certain tables. In the beginning we considered triangular numbers and a little later, square numbers. What would "pentagonal numbers" be? (You remember that a pentagon is a five-sided figure - the shape of the Pentagon in Washington.) Consider the following figure which is a set of pentagons:



We call 1 the first pentagonal number and 5 the next. In the next pentagon there will be 3 dots on a side and we add three sides with a total of 3 + 3 + 3 - 2 = 7 dots. (We subtract 2 for the vertices which we have counted twice.) The next time we would add 4 + 4 + 4 - 2, or 10 dots. Each time we add three more than we did the previous time. In this way we get the following table of pentagonal numbers:

Table VI

	1		5		12		22		35
first differences		4		7		10		13	
second differences			3		3		3		

From this and our previous experience it looks as if the formula which would fit this table would be of the form

$$an^2 + bn + c$$

for a proper choice of the numbers a, b, and c. Let us see if this will work out. Then the n-th and $(n + 1)$st terms would be

$$an^2 + bn + c \quad \text{and} \quad a(n + 1)^2 + b(n + 1) + c$$

and their difference would be

$$a \times [(n + 1)^2 - n^2] + b \times [(n + 1) - n] + c - c.$$

We have already found that $(n + 1)^2 - n^2 = 2n + 1$ and hence the difference is

$$a(2n + 1) + b = 2an + (a + b).$$

Now this has to be equal to the first difference formula. But we can find this. Since the first differences of the first differences are all 3, the formula for the first difference in the table must be $3n + r$ and r must be 1 to have it give the number 4 when $n = 1$. So we have

$$3n + 1 = 2an + (a + b).$$

This means that

$$2a = 3 \quad \text{and} \quad a + b = 1,$$

which gives $a = \frac{3}{2}$ and $b = -\frac{1}{2}$.

Hence the formula for the numbers in the first line of Table VI, the pentagonal numbers, should be

$$\frac{3}{2} n^2 - \frac{1}{2}n + c$$

for a proper choice of c. Putting $n = 1$ in the formula and setting it equal to the first entry, 1, in the table, we get

$$1 = \frac{3}{2} \times 1^2 - \frac{1}{2} \times 1 + c = 1 + c$$

which shows that c must be zero. So our formula for the n-th term in the first row of Table VI seems to be

$$\frac{3}{2}n^2 - \frac{1}{2}n = \frac{3n^2 - n}{2}$$

and trying values for n will show you that it does indeed fit. Furthermore it has to fit since the first differences are fixed and they determine the entries on the first line after the first entry is given.

Exercises 5-3-b

1. Find formulas which fit each of the following tables of values:

   (a) 2 7 12 17 22 27

   (b) 5 19 43 77 121

   (c) 8 20 38 62 92

2. What kind of a formula do you think would fit the following table of values:

   2 10 30 68 130 222?

3. Find the formula which will fit the numbers in problem 2.

4. Show that the following numbers are the hexagonal numbers (a hexagon is a six-sided figure).

   1 6 15 28 45

   Find a formula for the hexagonal numbers.

5. Use the methods of this section to find the formula for the sum of the first n squares.

6. Have you ever noticed cannon balls piled in a triangular pyramid on an old battlefield? There might be a little pile

with 3 in a triangle on the bottom and 1 on top of it, giving 4 in all. If there were three tiers, the triangle on the ground would have 6, plus the four above, would be 10. If there were four tiers, there would be 10 on the bottom with a total of 20 in the pile. These numbers are called pyramidal numbers and are

$$1 \quad 4 \quad 10 \quad 20 \quad 35 \quad \cdots$$

Can you discover any relationship between them and the triangular numbers? Can you find a formula for the pyramidal numbers?

7. Suppose there is a table of values in which the third differences form an arithmetic progression. Can you guess what sort of a formula would fit the numbers of the table?

8. There is a famous theorem that every integer can be expressed as the sum of three or fewer triangular numbers. Try it out: $1 = 1$, $2 = 1 + 1$, $3 = 3$, $4 = 1 + 3$, $5 = 1 + 1 + 3$, $\cdots$, $14 = 1 + 3 + 10$, etc.
   Notice that the numbers 5 and 14 actually need to have three triangular numbers in the sum. The theorem also says that every integer which is positive can be expressed as the sum of four or fewer square numbers, five or fewer pentagonal numbers, etc. You might be interested in trying this out. The proof is very difficult.

9. There are some sets of numbers that have the property that no row of differences, no matter how far you go, form an

arithmetic progression. Two such sets are

(a) $2 \quad 2^2 \quad 2^3 \quad 2^4 \quad \cdots \quad 2^n \quad \cdots$

(b) $1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad \cdots$

where in the second sequence each number is the sum of the previous two. Show that no matter how many differences you take, no set will form an arithmetic progression.

10. We know from problem 10 in section 1, that any two given numbers may be used to start an arithmetic progression. Why does this show that no matter what two numbers you may name, I can find a formula like: $an + b$ which has these two numbers as values for $n = 1$ and $n = 2$?

11. Look at problem 7 in section 2 and see if you can answer the following question: Given any three numbers, we can find a formula like

$$an^2 + bn + c$$

which will have the given numbers as values when $n = 1$, $n = 2$, $n = 3$?

12. What kind of a formula do you think would fit any set of four values? Can you draw any general conclusions?

SUPPLEMENTARY UNIT 6

THE LATEST DOPE ON PRIMES

6-1.

We are going to report to you on results published by Professor Raphael M. Robinson, of the University of California at Berkeley, in October, 1958, issue of the Proceedings of the American Mathematical Society. This will give you some idea of how research mathematicians are applying high-speed computers to solve problems about primes.

Robinson's note is based on calculations carried out during 1956 and 1957 on the SWAC (Standards Western Automatic Computer) at the University of California in Los Angeles.

To obtain an idea of the meaning of this work, let us think for a moment about the problem of finding out whether a given number. n is a prime. According to the definition of a prime, we must find out whether n is divisible by some smaller number other than 1. The most obvious method is to divide n by the numbers $2, 3, 4, \cdots$, up to $n - 1$. If any of these numbers divide evenly into n, then n is not a prime. If none of these divisions come out evenly, then n is a prime. This method requires $n - 2$ divisions. If n is about 10^{100}, and if each division requires .001 seconds, then this would take about 10^{97} seconds. How many seconds are there in a year? About how many years would this take?

We could shorten the work very much if we think a little. If n is not a prime, then n can be expressed as a product of two smaller numbers:

$$n = a \cdot b.$$

If a is the smaller of these factors, then n is at least $a \cdot a = a^2$.

$$n \geq a^2.$$

Therefore, if n is not a prime, then it is divisible by some number a whose square is at most n. To test whether n is a prime, it is enough to divide n by the numbers $2, 3, \cdots$, up to the largest number whose square is no larger than n. If $n \leq 1{,}000{,}000$, then we do not have to try any divisors greater than 1,000, since $1{,}000^2 = 1{,}000{,}000$. Thus to see whether 999,997 is a prime, we only need to divide by $2, 3, \cdots, 999$. By this method we only need 998 divisions instead of 999,995 divisions in the previous method.

If n is about 10^{100}, then this method requires only (!!) about 10^{50} divisions, for $10^{50} \cdot 10^{50} = 10^{100}$. If each division takes .001 seconds, how many years would it take by this method to test whether n is a prime.

If we wish to test really large numbers, we must look for better methods so that we can obtain the answers in a reasonable time. Therefore, mathematicians try to find special classes of numbers which have special properties which enable us to reduce the work even more.

For example, a great deal of work has been done on numbers which are one less than a power of 2. We may represent such numbers in the form

$$n = 2^m - 1.$$

If $m = 2$, then $n = 2^2 - 1 = 4 - 1 = 3$, which is a prime. If $m = 4$, then $n = 2^4 - 1 = 16 - 1 = 15$, which is not a prime. If m is not a prime, then n cannot be a prime. But m may be a prime without n being a prime.

Exercises 6-1

1. Make a table up to $m = 20$:

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14-20
n	1	3	7	15	31									

2. Test the statements

   If m is divisible by 2, then n is divisible by 3.

   If m is divisible by 3, then n is divisible by 7.

   If m is divisible by 5, then n is divisible by 31.

   What is the general law?

Robinson reports on numbers which are one more than a small multiple of a power of 2, that is, numbers of the form

$$n = (k \cdot 2^m) + 1,$$

where k is a small odd number.

He and his group tested for primeness all numbers of this form with $k < 100$ and $m < 512$, as well as a few larger numbers. First they divided by all numbers less than 10,000; and for $k \leq 7$ they tried divisors up to 100,000. After eliminating all small factors in this way, they then applied a theorem stated by Proth in 1878. Let us see if we can't get some idea of what Proth's theorem says and how it is used without trying to examine all of the details.

Proth's theorem gives a method of testing numbers of the form $n = (k \cdot 2^m) + 1$ for primeness provided the counting number k is odd and less than 2^m. We can avoid much of the complication of the statement of Proth's theorem if we restrict ourselves to the case where k is not divisible by 3. Thus we may use

$$k = 1, 5, 7, 11, 13, 17, \cdots$$
$$m = 1, 2, 3, 4, 5, 6, 7, \cdots$$

and we are able to test the numbers $n = (k \cdot 2^m) + 1$ for primeness. For these numbers n Proth's theorem states that

6.2 n _is prime if and only if it is a factor of_ $3^{\frac{n-1}{2}} + 1$.

Does this look mysterious to you? It is likely that it does, because you are not a mathematician. It would very probably look a bit mysterious even to a mathematician if he didn't happen to be familiar with the special techniques which are needed for a proof of this particular theorem. However, if you will accept our word that it is a true theorem (and a great many very respectable mathematicians will testify to its being true) then it shouldn't be hard to see what it _says_ and how it is _used_.

In the first place, what does $3^{\frac{n-1}{2}} + 1$ mean? The expression $\frac{n-1}{2}$ is being used as an exponent. The number n we are using here is odd. (Why? What is the form of n?) Thus n - 1 is even, so that $\frac{n-1}{2}$ is a counting number. Thus $3^{\frac{n-1}{2}} + 1$ is just one more than 3 raised to a counting number power. To test n for primeness we need only find this number and then divide it by n. If this division comes out even then n is a prime; otherwise n is a composite.

What numbers can we test for primeness by this method? Let us list a few of them in a table and then apply the test to some of them. Fill in the blank spaces in the table below. Remember that Proth's theorem requires that $0 < k < 2^m$, and that we have restricted ourselves to numbers k which are not divisible by 3.

k	m	$n = (k \cdot 2^m) + 1$
1	1	3
1	2	5
1	3	9
5	3	
7	3	57
1	4	
5	4	81
7	4	113
11	4	
13	4	209
1	5	33

k	m	$n = (k \cdot 2^m) + 1$
5	5	161
7	5	225
11	5	
1		65
	6	321
	5	417
	5	545
		2,817
		17,409
17		2,177
	10	7,169
		10,241

Now let us see how the test works for a few of these numbers. To refresh our memories we restate it here:

If $n = (k \cdot 2^m) + 1$ where $0 < k < 2^m$ and k is not divisible by 3 then n is prime if and only if it is a factor of

$$3^{\frac{n-1}{2}} + 1.$$

Example 1: Let $k = 1$ and $m = 2$ so that $n = 5$. (Look it up in the table.) We are testing 5 for primeness. In this case $\frac{n-1}{2}$ is $\frac{4}{2}$ or 2, so

$$3^{\frac{n-1}{2}} + 1 = 3^2 + 1 = 9 + 1 = 10.$$

Is n a factor of $3^{\frac{n-1}{2}} + 1$? Is 5 a factor of 10? Yes, it is, so the test tells us that 5 is a prime. Does this check with what you already know?

Example 2: Let $k = 1$ and $m = 3$ so that $n = 9$. (Look it up.) We divide

$$3^{\frac{n-1}{2}} + 1 = 3^4 + 1 = 81 + 1 = 82$$

by 9. The division does not come out even, so the test tells us that 9 is not a prime. Does this check with what you already know about 9?

Example 3: If $k = 1$ and $m = 6$ then what is n? The table should tell you that $n = 65$. If it doesn't, work it out again. $\frac{n-1}{2}$ is 32, then, so

$$3^{\frac{n-1}{2}} + 1 = 3^{32} + 1 = 1,853,020,188,851,842.$$

We would have to divide this number by 65 to continue the test. It would not be worth the effort, however, since we can easily recognize that 65 has 5 as a factor, and is therefore not a prime.

Example 4: Let $k = 7$ and $m = 4$ so that $n = (k\cdot 2^m) + 1 = 113$. In this case the number $3^{\frac{n-1}{2}} + 1 = 3^{66} + 1$ is 9 times the square of 1,853,020,188,851,842 plus 1. If you are ambitious you may calculate this number and divide it by $n = 113$. The division will come out even if you do your work correctly, so what do you conclude about 113?

Exzmples 3 and 4 should convince us of one thing. Proth's theorem is not well suited for testing large numbers for primeness by hand calculation. However, large computers are constructed expressly to make calculations of the order of the ones which discouraged us above. And they do them quickly! On the SWAC the time for the test was no more than $1\frac{1}{2}$ minutes as long as $m < 512$. For m about 1000 and $k = 3, 5,$ or 7 the test took about 7 minutes. The number $n = (7\cdot 2^{1000}) + 1$ is larger than 10^{300}. Compare 7 minutes with the time it would take the machine to test 10^{300} for primeness by trying all possible factors. Earlier in this section you got some idea of this time for numbers of the order of 10^{100}.

For $k = 1$ the test had previously been carried out for all $m < 8192$, and the only primes of this form which have been found are the cases

$$m = 0, 1, 2, 4, 8, \text{ and } 16.$$

The largest new prime discovered by this work is the case $k = 5$, $m = 1947$:

$$n = (5 \cdot 2^{1947}) + 1.$$

If you wish to estimate this number, first notice that

$$10^3 = 1000 < 2^{10} = 1024.$$

Therefore we have

$$2^{1947} > 2^{1940} = (2^{10})^{194} > (10^3)^{194} = 10^{582}.$$

Therefore n has more than 582 digits. On the other hand, notice that

$$2^{13} = 8096 < 10^4.$$

Therefore we have

$$n < 1 + (8 \cdot 2^{1947}) = 1 + (2^3 \cdot 2^{1947})$$
$$= 1 + 2^{1950} = 1 + (2^{13})^{150}$$
$$< 1 + (10^4)^{150} = 1 + 10^{600}.$$

Consequently n has no more than 600 digits.

Remember that by using the theorem of Proth, this prime was discovered by a single division taking a matter of minutes. By using either of the cruder methods discussed before at least 10^{291} divisions would have been necessary. How long would this have taken at the rate of a thousand divisions per second?

This number is the fourth largest prime known at present. The larger ones are the numbers

$$n = 2^m - 1$$

with $m = 3217$, 2281, and 2203. The latter two were reported by

Robinson in the Proceedings of the American Mathematical Society in 1954. The largest one was reported early in 1958 by H. Riesel in Mathematical Tables and Aids to Computation (page 60).

Example 5: Estimate the number of digits in each of three primes.

* Perhaps you would be interested in the general statement of Proth's theorem. For numbers $n = (k \cdot 2^m) + 1$ with k *divisible* by 3 the important difference in the test for primeness is that the number $3^{\frac{n-1}{2}} + 1$ must be replaced by a new number. The number to use is of the form

$$a^{\frac{n-1}{2}} + 1$$

where a is a counting number which may have to be chosen differently for different values of k and m. The condition which a must satisfy will be found in the statement of Proth's theorem.

Theorem: Let $0 < k < 2^m$ and $n = (k \cdot 2^m) + 1$. Suppose a is a counting number which has the property: *no* sum of a and a multiple of n is a perfect square. (Alternative: the sum of a and a multiple of n is *never* a perfect square.)

Then n is a prime if and only if it is a factor of

$$a^{\frac{n-1}{2}} + 1.$$

The condition which a must satisfy is rather a strange one. It would seem that it might be difficult to find a number which satisfies it in some cases. We could never find such a number by any number of trial operations, for the condition which a must satisfy involves a statement about all multiples of n. We may reject some choices of a on the basis of a single calculation, though. If $k = 3$, and $m = 2$ so that $n = 3 \cdot 2^2 + 1 = 13$ then would $a = 4$ do? No, because $117 + a = 117 + 4 = 121$ is a perfect square, and 117 is a multiple of $n = 13$. To find a number a which we can be sure will fit the condition for a given n, then, we will have to use reasoning. We will have to reason that, for a certain number a, no matter how many multiples of n we try, adding a will never give a perfect square. Mathematicians know enough about numbers so that finding such a number is not a very difficult problem. As you may have guessed from the discussion above, it is possible to show that whenever k is not divisible by 3 the number $a = 3$ satisfies the condition of the theorem. Once we have found the right number a to go with n we can avoid the many tedious calculations necessary to test a large number for primeness. Instead of dividing n by all prime numbers whose squares are less than n, we need only perform one calculation. We simply try the division

$$\left(a^{\frac{n-1}{2}} + 1\right) \div n;$$

if it comes out even n is a prime, if not, n is not a prime.